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A Note on Ultrafilter Selectivity

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Here we discuss the relation between the ultrafilter selectivity and other combinatorial filter properties, i. e. regularity descending completness and weak normality. Selectivity is important for characterizations in Rudin Keisler order, though little is known about the existence of ultrafilters with this kind of property. We begin with some well known definitions and notation, taken majorly from [2]. The considered ultrafilters are over cardinals, which does not reduce generality.

Definition. An ultrafilter D over k is λ-uniformly selective iff for every $f\in{}^k K$, there is $A\in D$ such that either $f|A$ is one-to-one or $|f[A]|<k$.

If D is ω-uniformly selective then D is simply selective and if D is K uniformly selective then D is uniformly selective.

The next two results, from [2], illustrate the use of this concept.

Lemma 1. *Let $\omega\leqq\lambda\leqq k$ and let D be an uniform ultrafilter on k. The following are equivalent:*

1) D is λ-uniformly selective;

2) Rudin Keisler type of D, $\tau(D)$ is minimal in Rudin Keisler order of $\tau(U_\lambda(k))$, (i. e. D is RK minimal type in the set of all ultrafilters over k with the norm greater that or equal to λ).

Lemma 2. *Let $\omega\leqq\lambda\leqq k$ and let D be a λ − uniformly selective, λ − complete uniform ultrafilter on k. Then:*

1) D is k-complete (and k is a measurable cardinal); and

2) if $k>\omega$ then D is type equivalent to a normal ultrafilter on k.

From Lemma 1 it follows that if D is selective then it is RK type-minimal in the set of all ultrafilters (types) over the index k. In this case the type of D coincides with a type of a normal ultrafilter. Also, if D is uniformly selective, then it's type is RK minimal in the set of uniform ultrafilters over the given index. In this case D is type equivalent to a weakly normal ultrafilter.

The following notions we need later.

Definition. Ultrafilter D is λ-descendingly complete iff every descending chain of λ element of D has intersection in D. If D is not λ-descendingly complete it is said to be λ-descendingly incomplete.

Definition. Uniform ultrafilter D over k is (α, β) regular, if D contains a family of cardinality β whose every α elements have empty intersection. It is well known that: D is conutably incomplete iff it is (ω, ω) regular; D is (α, α) regular if D is $(cf\alpha, cf\alpha)$ regular; D is (k, k) regular; if D is (α, β) regular and $\alpha \leqq \alpha' \leqq \beta' \leqq \beta$ then D is (α', β') regular. Regularity trace for D is defined by regtr $(D) = \{(\alpha, \beta) \mid D \text{ is } (\alpha, \beta) \text{ regular}\}$. It is clear from the definition that if D is λ uniformly selective then for all $\lambda' > \lambda$, D is λ'-uniformly selective.

Selecitivity properties link in a simple way to notations of normal and weakly normal ultrafilters whith are distant in consistency strength.

The next theorem [3] relates the above notions.

Theorem 3. 1) *For regular λ, (λ, λ) regularity is equivalent to λ-descending incompleteness;*

2) *From (an instance) of strong compactness follows the consistency of existence of an ultrafilter (over 2^{ω}) which is (λ, λ) regular and λ-descendingly complete for a singular cardinal λ of cofinality ω_1.*

The former enables us to conclude

Theorem 4. *Let D be an uniform ultrafilters over k. If D is λ-uniformly selective, then for all regular cardinals α, such that $\lambda \leqq \alpha < k$, D is not (α, α) regular (or, equivalently, D is α-descendingly complete).*

P r o o f. Let D be (α, α) regular for a regular cardinal α, $\lambda \leqq \alpha < k$. Then D is α descendingly incomplete and the sequence β_D, $\beta < k$ is not cofinal in the ultrapower $\prod_D \alpha$, as noticed in [1], that can be easily checked. Next, if $f_D \in \prod_D \alpha$ and f_D is beyond all constants then $|\text{rng}(f)| = \alpha$, modulo D, i. e. there is $A \in D$ such that $|f[A]| = \alpha$. For, otherwise, there would be a $B \in D$ such that $|f[B]| < \alpha$. Then, since f is cofinal in α, α is singular. Thus, $|f[A]| = \alpha$ for same $A \in D$, and D is not λ-selective.

The proof is not applicable for α-singular, which leaves unresolved the question (for uniform ultrafilter D over K):

5. If α is singular cardinal and f cofinal in α modulo D, (i. e. f_D is beyond constants in $\prod_D \alpha$), is there an $A \in D$ such that $|f[A]| = cf\alpha$?

Positive answer to 5. simplifies the situation. Then regularity trace of an ultrafilter provides simple test: if it is λ uniformly selective and for which λ, especially those between ω and the index. The examples for such filters, starting with the saturated filter of Silver could be obtained as in [3].

The negative answer to the above question might reduce the existence of selective ultrafilters.

We observe that lemma 2 can be improved as

Lemma 2'. *Let $\omega \leqq \lambda \leqq k$ and D be a λ-uniformly selective, ω_1-complete ultrafilter on k*

1) *D is k-compelete, selective ultrafilter (and k is measurable)*

2) *if* $k>\omega$ *then D is type equaivalent to a normal ultrafilter on k.*
The following theorem is from [2].

Theorem 6. *Let* $\alpha \geqq \omega$ *and let D be a uniform ultrafilter on* α^+. *Then either a) D is* α*-descendingly incomplete or* b) *there is a regular cardinal* $\beta \leqq \alpha$ *such that D is* (β, α^+)*-regular.*
Combining it with 5. and Theorem 4. we can conclude

Corollary 7. *If D is an uniform ultrafilter over a successor cardinal k, then for every regular cardinal* λ, $\omega \leqq \lambda < k$, *D is not* λ*-uniformly selective.*

In [4] we discuss another property, generalized normality, apparently similar to λ-uniform selectivity.

Definition. An ultrafilter D over k is λ-weakly normal iff there is $f \in k^\lambda$ such that f_D is beyond all constants in $\prod_D \lambda$, and f_D is minimal.

If a uniform ultrafilter D over k is λ-weakly normal then it generates a minimal element in Rudin Keisler order at the level of ultrafilters of norm λ, below D. The consistency of existence of ultrafilters with this property over an index $\leqq 2^\omega$ is proved relative consistency of (an instance) of strong compactness (in [4]).

How are these properties related?

Clearly, if D over k is λ-weakly normal, then it is not λ^+-uniformly selective. What for cardinals greater than λ^+? The examples in [4] of λ-weakly normal ultrafilters over k ($\leqq 2^\omega$) is obtained from λ-complete, (λ, k)-regular ultrafilter in CCC extension, by expansion of original ultrafilter, which therefore contains a (k, λ) regular family, so the obtained λ-weakly normal ultrafilter is not α-uniformly selective for any regular cardinal α. A variant for a singular cardinal α in case that 5. is true can be constructed.

The opposite direction, i. e. if λ-uniform selectivity of D implies its α-weak normality, for some α remains unknown.

This provides the following

Theorem 7. *It is consistent (relative strong compactness) that there is an ultrafilter D (over* 2^ω*) which is* λ*-weakly normal for some* $\lambda < 2^\omega$ *and which is not* α*-uniformly select for any regular* $\alpha < 2^\omega$.

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