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On a Relation between the Two-Dimensional H-Transforms in Terms of Erdélyi-Kober Operators

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A relationship between the two-dimensional integral H-transforms is established in terms of the Weyl-type two-dimensional Erdélyi-Kober operators of fractional differintegration. Special cases of this result yield relations concerning the two-dimensional Laplace, Whittaker and G-transforms.

1. Introduction

This paper is intended to go on further with a long-established trend on the borderline of Integral Transforms and Fractional Calculus. It is well-known that the Riemann-Liouville and Weyl operators of fractional integration (differentiation) carry out interesting relationships between the classical integral transforms and some of their generalizations as well as between integral transforms of one and the same kind but with different indices. More general operators of the Fractional Calculus like the Erdélyi-Kober operators or compositions of them have been recently used in finding new results relating integral transforms of rather a general nature. The two-dimensional case is considered here.

As far as we are dealing with a generalized form of the so-called Weyl fractional integral, we follow the tradition of considering a sufficient class of "good functions" descending from M. J. Lighthill [10, p. 15]. Modifying Miller's definition [13, p. 82] for the two-dimensional case, let us denote by $\mathscr{A}$ the class of the functions $f(x, y)$ which are differentiable any number of times and have their partial derivatives behaving as $O(|x|^{-\varepsilon_1}, |y|^{-\varepsilon_2})$ with arbitrary $\varepsilon_1 > 0, \varepsilon_2 > 0$ when $x \to \infty$, $y \to \infty$ (cf. [15]). Examples of "good functions" in this sense are provided by the functions $\mathscr{T}(x, y) \exp(-\gamma x - \delta y)$, where $\gamma > 0, \delta > 0$ and $\mathscr{T}(x, y)$ is a polynomial of x, y.

The two-dimensional Erdélyi-Kober operator of fractional integration of orders $\alpha > 0$, $\beta > 0$ is usually defined in the class $\mathscr{A}$ as follows:

$$(1.1) \qquad K_x^{\eta, \alpha} K_y^{\delta, \beta} f(x, y)$$

$$= \frac{x^\eta y^\delta}{\Gamma(\alpha)\Gamma(\beta)} \int_x^\infty \int_y^\infty u^{-\eta-\alpha} v^{-\delta-\beta} (u-x)^{\alpha-1} (v-y)^{\beta-1} f(u, v)\, du\, dv,$$

where η, δ are real numbers. More generally, an Erdélyi-Kober operator of Weyl-type in two variables is defined by the differintegral expression

$$(1.2)\qquad K_x^{\eta,\alpha}K_y^{\delta,\beta}f(x,y)=\frac{(-1)^{m+n}x^{\eta}y^{\delta}}{\Gamma(m+\alpha)\,\Gamma(n+\beta)}$$

$$\times D_{x,y}^{m+n}\{\int_x^{\infty}\int_y^{\infty}u^{-\eta-\alpha}v^{-\delta-\beta}(u-x)^{m+\alpha-1}(v-y)^{n+\beta-1}f(u,v)\,du\,dv\}$$

for arbitrary real (complex) α and β, where $m, n=0, 1, 2, \ldots$ and $D_{x,y}^{m+n}$ stands for the partial differentiation $\partial^{m+n}/\partial x^m\partial y^n$. For $f(x, y)\in\mathscr{A}$ this differintegral exists and also belongs to $\mathscr{A}$ (the arguments are the same as in [13, p. 82]). In particular, if $\alpha<0$, $\beta<0$ and m, n are positive numbers such that $m+\alpha>0$, $n+\beta>0$, then (1.2) yields the partial fractional derivatives of $f(x, y)$. For $\eta=-\alpha$, $\delta=-\beta$ we receive the two-dimensional Weyl operators (R. K. Raina and V. S. Kiryakova [15]), namely:

$$(1.3)\qquad (-1)^{\alpha+\beta}\,W_x^{\alpha}W_y^{\beta}f(x,y)=\frac{(-1)^{m+n}}{\Gamma(m+\alpha)\,\Gamma(n+\beta)}$$

$$\times D_{x,y}^{m+n}\{\int_x^{\infty}\int_y^{\infty}(u-x)^{m+\alpha-1}(v-y)^{n+\beta-1}f(u,v)\,du\,dv\}.$$

Further generalizations of the Fractional Calculus' operators W_x^{α}, $K_x^{\eta,\alpha}$, their compositions and their Riemann-Liouville counterparts R_x^{α}, $I_x^{\eta,\alpha}$ have been investigated by V. S. Kiryakova [8], [9] by introducing the Meijer's G-functions and the more general Fox's H-functions as kernel-functions. Further details and development of this Generalized Fractional Calculus can be found in S. L. Kalla and V. S. Kiryakova [6], [7].

Here we use the Fox's H-function as kernel-function of a generalized integral transform in two dimensions whose relationship with the Erdélyi-Kober operators $K_x^{\eta,\alpha}K_y^{\delta,\beta}$ is to be established. This special function of hypergeometric type and quite a general nature, depending on the "orders" M, N, P, Q and $2(P+Q)$-parameters (a_P), (A_P), (b_Q), (B_Q), can be defined in terms of a Mellin-Barnes type contour integral ([5], [12]):

$$(1.4)\qquad H_{P,Q}^{M,N}(x)=H_{P,Q}^{M,N}[x|{(a_P,A_P)\atop (b_Q,B_Q)}]$$

$$=H_{P,Q}^{M,N}[x|{(a_1,A_1),\ldots,(a_P,A_P)\atop (b_1,B_1),\ldots,(b_Q,B_Q)}]=\frac{1}{2\pi i}\int_{\mathscr{L}}\mathscr{H}(s)x^s\,ds,$$

with

$$\mathfrak{X}(s)=\frac{\prod_{h=1}^{M}\Gamma(b_h-B_hs)\prod_{j=1}^{N}\Gamma(1-a_j+A_js)}{\prod_{h=M+1}^{Q}\Gamma(1-b_h+B_hs)\prod_{j=N+1}^{P}\Gamma(a_j-A_js)}.$$

Here $0\leqq N\leqq P, 1\leqq M\leqq Q$ are nonnegative integers; A_j and B_h are positive; a_j and b_h are real (complex) numbers; $j=1, 2,\dots, P$, $h=1, 2,\dots, Q$ such that

$$A_j(b_h+v)\neq B_h(a_j-\lambda-1);\quad \begin{array}{l} v,\ \lambda=0,\ 1,\ 2,\dots; \\ h=1,\dots,M;\ j=1,\dots,N; \end{array}$$

$\mathscr{L}$ is a suitable contour separating the simple poles of the integrand $\mathfrak{X}(s)$ in (1.4). More details about the H-functions and their excellent properties can be found e. g. in A. M. Mathai and R. K. Saxena [12].

Definition. Under two-dimensional H-transform $\mathscr{H}(p, q)$ of a function $F(x, y)$ we mean the following repeated integral involving two different H-functions:

$$\text{(1.5)}\qquad \mathscr{H}(p,q)=\mathscr{H}_{P,Q,P_1,Q_1}^{M,N,M_1,N_1}[F(x,y);\ \rho,\sigma;\ p,q]$$

$$=\int_b^\infty\int_d^\infty (px)^{\rho-1}H_{P,Q}^{M,N}[(px)^k\,|\,{}^{(a_P,A_P)}_{(b_Q,B_Q)}](qy)^{\sigma-1}H_{P_1,Q_1}^{M_1,N_1}[(qy)^l\,|\,{}^{(\gamma_{P_1},C_{P_1})}_{(\delta_{Q_1},D_{Q_1})}]F(x,y)\,dx\,dy.$$

We assume that $\mathscr{H}(p, q)$ exists and belongs to $\mathscr{A}$; $b>0$, $d>0$,

$$\text{(1.6)}\qquad |\arg p^k|<\frac{1}{2}\pi\Phi_1,\ |\arg q^l|<\frac{1}{2}\pi\Phi_2,$$

where

$$\Phi_1=\sum_{j=1}^{N}A_j-\sum_{j=N+1}^{P}A_j+\sum_{h=1}^{M}B_h-\sum_{h=M+1}^{Q}B_h>0,$$

$$\Phi_2=\sum_{j=1}^{N_1}C_j-\sum_{j=N_1+1}^{P_1}C_j+\sum_{h=1}^{M_1}D_h-\sum_{h=M_1+1}^{Q_1}D_h>0$$

and

$$\sum_{h=1}^{Q}B_h-\sum_{j=1}^{P}A_j\geqq 0,\ \sum_{h=1}^{Q_1}D_h-\sum_{j=1}^{P_1}C_j\geqq 0.$$

Due to the great power of generality of the kernel H-functions, integral transforms (1.5) turn out to generalize a number of integral transforms like the two-dimensional Laplace, Whittaker, G-transforms. That is why the result we are proposing here provides extensions of the results of R. K. Saxena, O. P. Gupta and R. K. Kumbhat [18], A. K. Arora, R. K. Raina and C. L. Koul [1], R. K. Saxena and J. Ram [19], K. Nishimoto and R. K. Saxena [14], R. K. Raina and V. S. Kiryakova [15] and some other authors.

2. Relationship between the H-transforms via Erdélyi-Kober operators

Together with H-transforms (1.5) we consider the H-transform with the M-, P- and Q-orders increased by 1, namely:

$$(2.1)\qquad H_1(p, q)=\mathscr{H}^{M+1, N, M_1+1, N_1}_{P+1, Q+1, P_1+1, Q_1+1}[F(x, y);\ \rho, \sigma;\ p, q]$$

$$=\int_b^\infty\int_d^\infty (px)^{\rho-1}(qy)^{\sigma-1} H^{M+1, N}_{P+1, Q+1}\left[(px)^k \,\middle|\, {(a_P, A_P), (\alpha+\eta-\rho+1, k) \atop (\eta-\rho+1, k), (b_Q, B_Q)}\right]$$

$$\times H^{M_1+1, N_1}_{P_1+1, Q_1+1}\left[(qy)^l \,\middle|\, {(\gamma_{P_1}, C_{P_1}), (\beta+\delta-\sigma+1, l) \atop (\delta-\sigma+1, l), (\delta_{Q_1}, D_{Q_1})}\right] F(x, y)\,dx\,dy\ ,$$

provided that $\mathscr{H}_1(p, q)$ exists and belongs also to $\mathscr{A}$ and the other conditions on the parameters (including additional parameters $\alpha, \beta, \eta, \delta$) correspond to these in (1.6).

Theorem. *Let $\mathscr{H}(p, q)\in\mathscr{A}$ and $\mathscr{H}_1(p, q)$ be the two-dimensional H-transforms defined by* (1.5), (2.1) *under conditions of the form* (1.6) *and let $\alpha>0$, $\beta>0$. Then the following relationship between them, carried out by the two-dimensional Erdélyi-Kober operator* (1.1), *holds:*

$$(2.2)\qquad K_p^{\eta, \alpha} K_q^{\delta, \beta}\{\mathscr{H}(p, q)\}=\mathscr{H}_1(p, q),$$

where $\mathscr{H}_1(p, q)$ also belongs to $\mathscr{A}$.

Proof. In view of (1.1) and (1.5) the left-hand side of (2.2) has the form

$$K_p^{\eta, \alpha} K_q^{\delta, \beta}\{\mathscr{H}(p, q)\}=\frac{p^\eta q^\delta}{\Gamma(\alpha)\Gamma(\beta)}\int_p^\infty\int_q^\infty u^{-\eta-\alpha}v^{-\delta-\beta}(u-p)^{\alpha-1}(v-q)^{\beta-1}$$

$$\times\mathscr{H}(u, v)\,du\,dv=\frac{p^\eta q^\delta}{\Gamma(\alpha)\Gamma(\beta)}\int_p^\infty\int_q^\infty u^{-\eta-\alpha}v^{-\delta-\beta}(u-p)^{\alpha-1}(v-q)^{\beta-1}$$

$$\times\left\{\int_b^\infty\int_d^\infty (ux)^{\rho-1}(vy)^{\sigma-1}H^{M, N}_{P, Q}\left[(ux)^k \,\middle|\, {(a_P, A_P) \atop (b_Q, B_Q)}\right]H^{M_1, N_1}_{P_1, Q_1}\left[(vy)^l \,\middle|\, {(\gamma_{P_1}, C_{P_1}) \atop (\delta_{Q_1}, D_{Q_1})}\right]dx\,dy\right\}du\,dv.$$

Then, under conditions (1.6), a change in the order of the u, v- and x, y-integrals is permissible which is leading to the possibility to evaluate the inner u- and v-integrals. To this end we need the following result for the Weyl-type fractional integrals of order μ of Fox's H-function, viz.

$$(2.3)\qquad \frac{1}{\Gamma(\mu)}\int_p^\infty x^{-\lambda}(x-p)^{\mu-1}\, H_{P,Q}^{M,N}\left[(\omega x)^k \,\Big|\, {(a_P,\ A_P) \atop (b_Q,\ B_Q)}\right] dx$$

$$= p^{\mu-\lambda}\, H_{P+1,Q+1}^{M+1,N}\left[(\omega x)^k \,\Big|\, {(a_P,\ A_P),\ (\lambda,\ k) \atop (\lambda-\mu,\ k),\ (b_Q,\ B_Q)}\right],$$

provided $\mu>0$, $(\lambda-\mu+\max(1-a_j/A_j))>0$, $j=1,2,\dots,N$; $|\arg\omega|<\frac{1}{2}\pi C$, where

$$C=\sum_{j=1}^{N} A_j-\sum_{j=N+1}^{P} A_j+\sum_{h=1}^{M} B_h-\sum_{h=M+1}^{Q} B_h>0;\ \sum_{h=1}^{Q} B_h-\sum_{j=1}^{P} A_j\geq 0.$$

Integral (2.3), on its turn, can be established by means of the formula (see [4, p. 201, eq. 6]):

$$(2.4)\qquad \int_p^\infty x^{-\lambda}(x-p)^{\mu-1}\,dx=p^{\mu-\lambda}\frac{\Gamma(\mu)\,\Gamma(\lambda-\mu)}{\Gamma(\lambda)},\ \text{where } 0<\mu<\lambda.$$

After this procedure we reach to the right-hand side of (2.2) and the theorem is proved. As far as the two-dimensional Weyl-type Erdélyi-Kober operator $K_x^{\eta,\alpha}K_y^{\delta,\beta}$ preserves the class $\mathscr{A}$, it follows that $\mathscr{H}_1(p,q)$ also belongs to $\mathscr{A}$.

Let us note that the statement of the theorem can be easily extended for arbitrary real α, β by using definition (1.2) for the Erdélyi-Kober operators and differentiating under the signs of the integrals.

3. Special cases

First, let us note that in the one-dimensional case the problem of connecting the H-transforms by means of Weyl-type fractional integrals was stated and solved by R. K. Raina and C. L. Koul [17]. Their result leads to many particular cases concerning the Laplace, Stieltjes and Whittaker one-dimensional transforms.

The two-dimensional case considered here is also rich in interesting special cases.

Thus, using the identity

$$(3.1)\qquad H_{1,2}^{2,0}\left[x\,\Big|\,{(1-\lambda,\ 1) \atop (\frac{1}{2}+\mu,\ 1),\ (\frac{1}{2}-\mu,\ 1)}\right]=\exp\left(-\frac{1}{2}x\right)W_{\lambda,\mu}(x),$$

where $W_{\lambda,\mu}(x)$ is the Whittaker confluent hypergeometric function ([20, p. 340], see also [3, §6.9]) with $(\frac{1}{2}-\lambda+\mu)>0$, $x>0$:

$$(3.2)\qquad W_{\lambda,\mu}(x)=\frac{\exp(-\frac{1}{2}x)x^{\mu+\frac{1}{2}}}{\Gamma(\frac{1}{2}-\lambda+\mu)}\int_0^\infty t^{-\lambda-\frac{1}{2}+\mu}(1+t)^{\lambda+\mu-\frac{1}{2}}\exp(-tx)\,dt,$$

we obtain from H-transform (1.5) the so-called Whittaker transform of two variables (R. K. Saxena and J. Ram [19]):

$$(3.3)\qquad \mathscr{H}_2(p,q)=\mathscr{W}^{\lambda,\mu}_{\lambda_1,\mu_1}[F(x,y);\ \rho,\sigma;\ p,q]$$

$$=\int_b^\infty\int_d^\infty (px)^{\rho-1}(qy)^{\sigma-1}\exp(-\frac{1}{2}px-\frac{1}{2}qy)\,W_{\lambda,\mu}(px)\,W_{\lambda_1,\mu_1}(qy)F(x,y)\,dx\,dy,$$

where $\mathrm{Re}(p)>0$, $\mathrm{Re}(q)>0$. So, for $M=M_1=2$, $N=N_1=0$, $P=P_1=1$, $Q=Q_1=2$, $a_1=1-\lambda$, $A_1=1$, $b_{1,2}=\frac{1}{2}\pm\mu$, $B_{1,2}=1$, $\gamma_1=1-\lambda_1$, $C_1=1$, $\delta_{1,2}=\frac{1}{2}\pm\mu_1$, $D_{1,2}=1$, $\mathscr{H}(p,q)=\mathscr{H}_2(p,q)\in\mathscr{A}$, $\alpha>0$, $\beta>0$, $\mathrm{Re}(p^k)>0$, $\mathrm{Re}(q^l)>0$ our theorem, i.e. (2.2), gives:

$$(3.4)\qquad K_p^{\eta,\alpha}K_q^{\delta,\beta}\{\mathscr{H}_2(p,q)\}$$

$$=\mathscr{H}^{3,0;3,0}_{2,3;2,3}[F(x,y);\ \rho,\sigma;\ p,q]=\mathscr{H}_3(p,q)$$

$$=\int_b^\infty\int_d^\infty (px)^{\rho-1}(qy)^{\sigma-1}H^{3,0}_{2,3}\left[(px)^k\left|\begin{matrix}(1-\lambda,1),(\alpha+\eta-\rho+1,k)\\(\eta-\rho+1,k),(\frac{1}{2}+\mu,1),(\frac{1}{2}-\mu,1)\end{matrix}\right.\right]$$

$$\times H^{3,0}_{2,3}\left[(vy)^l\left|\begin{matrix}(1-\lambda_1,1),(\alpha+\eta-\rho+1,l)\\(\delta-\sigma+1,l),(\frac{1}{2}+\mu_1,1),(\frac{1}{2}-\mu_1,1)\end{matrix}\right.\right]F(x,y)\,dx\,dy,$$

where $\mathscr{H}_3(p,q)$ exists and belongs to $\mathscr{A}$.

If we take now $k=l=1$, then (3.4) reduces to a result recently proved by Saxena and Ram [19, p. 28]. The cases $\rho=\mu+\frac{1}{2}$, $\sigma=\mu_1+\frac{1}{2}$ or $\rho=\eta+\lambda$, $\sigma=\delta+\lambda_1$ provide Corollaries 1, 2 ([19, p. 29]).

Next, taking $\rho=\sigma=1$ and using the identity

$$W_{m+\frac{1}{2}\pm m}(x)=x^{m+\frac{1}{2}}\exp(-\frac{1}{2}x),$$

we obtain from the Whittaker transform (3.3) the two-dimensional Laplace transform ([2]):

(3.5) $$\mathscr{L}(p, q)=\mathscr{L}[F(x, y);\ p, q]$$

$$=\int_0^\infty\int_0^\infty \exp(-px-qy)F(x, y)\,dx\,dy,\ \mathrm{Re}(p)>0,\ \mathrm{Re}(q)>0$$

of the function

$$F(x, y)=\begin{cases} f(a\sqrt{x^2-b^2}, c\sqrt{y^2-d^2}), & x>b>0, y>d>0,\\ 0, & \text{otherwise.}\end{cases}$$

Results concerning $\mathscr{L}(p, q)$ and $K_p^{\eta,\alpha}K_q^{\delta,\beta}$ have been obtained by R. K. Saxena, O. P. Gupta nd R. K. Kumbhat [18], which for $\eta=-\alpha$, $\delta=-\beta$ are reduced to the relation, due to A. K. Arora, R. K. Raina and C. L. Koul [1]. If moreover, $a=c=1$, $b=d=0$, then we obtain the relation between the two-dimensional Laplace transforms of functions $f(x, y)$ and $x^\alpha y^\beta f(x, y)$, expressed by means of the two-dimensional Weyl operators (1.3), found in R. K. Raina and V. S. Kiryakova [15, p. 1274], namely:

(3.6) $$(-1)^{\alpha+\beta}W_p^\alpha W_q^\beta\{\mathscr{L}[f(x, y);\ p, q)\}$$
$$=\mathscr{L}[x^\alpha y^\beta f(x, y);\ p, q],\ \mathrm{Re}(p)>0,\ \mathrm{Re}(q)>0.$$

Let us note also that in the case when all the parameters A_j, C_j, B_k, D_k of the H-functions in (1.5) are equal to 1, the kernel H-functions turn into Meijer's G-functions (see [3], [11]) because of the identity ([12, p. 10]):

(3.7) $$H_{P,Q}^{M,N}\left[X\left|\begin{matrix}(a_P, 1)\\(b_Q, 1)\end{matrix}\right.\right]=G_{P,Q}^{M,N}\left[x\left|\begin{matrix}(a_P)\\(b_Q)\end{matrix}\right.\right].$$

Then the theorem we have just proved turns into a result recently established by K. Nishimoto and R. K. Saxena [14, p. 25].

Other interesting specializations can be discussed too.

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