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A Generalization of the Bernstein Polynomials

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For the functions $f\in C^r[0, 1]$, $r=0, 1, 2,\dots$ the polynomials

$$B_{n,r}(f;x)=\sum_{k=0}^{n}\sum_{i=0}^{r}\frac{f^{(i)}(k/n)}{i!}(x-k/n)^i\binom{n}{k}x^k(1-x)^{n-k}$$

are introduced. For $r=0$ they coincide with the classical Bernstein polynomials, but for $r\geqq 1$ in contrast with the last ones, they are sensitive to the degree of smoothness of the function f as approximations to f.

As it is widely accepted, the n-th Bernstein polynomial for a function $f:[0, 1]\to R$ is said to be

$$(1)\qquad B_n(f;x)=\sum_{k=0}^{n}f(k/n)\binom{n}{k}x^k(1-x)^{n-k},\ n=0, 1, 2,\dots$$

By $C^r[0, 1]$ $(C^0[0, 1]=C[0, 1])$, $r=0, 1, 2,\dots$ we denote the set of all functions $f:[0, 1]\to R$, with a continuous derivative of order r on the segment $[0, 1]$.

The following theorems are classical results.

Theorem of S. N. Bernstein [1]. *Let* $f\in C[0, 1]$ *and* $B_n(f;\ x)$ *be the Bernstein polynomial for f. Then*

$$\lim_{n\to\infty}B_n(f;\ x)=f(x)$$

uniformly on $[0, 1]$.

The Bernstein polynomials possess many remarkable properties. We state only two of them.

Theorem of T. Popoviciu [2]. *Let* $f\in C[0, 1]$ *and* $B_n(f;\ x)$ *be the Bernstein polynomial for f. Then, for every* $x\in[0, 1]$ *the following inequality holds:*

$$(2)\qquad |f(x)-B_n(f;\ x)|\leqq\frac{3}{2}\omega(f;\ n^{-1/2})$$

with

$$\omega(f;\ s)=\sup\{|f(x)-f(y)|: x,\ y\in[0,\ 1],\ |x-y|\leqq s\}$$

($\omega(f;\ s)$ is the modulus of continuity of the function f in the segment [0, 1]).

Theorem of E. V. Voronovskaya [3]. *Let $f\in C^2[0,\ 1]$ and $B_n(f;\ x)$ be the Bernstein polynomial for f. Then for every $x\in[0,\ 1]$ the asymptotical equality*

$$B_n(f;\ x)=f(x)+\frac{f''(x)}{2n}x(1-x)+\frac{\rho_n}{n} \tag{3}$$

with

$$\lim_{n\to\infty}\rho_n=0$$

is exact.

From the Voronovskaya's theorem follows that no further improvement of the function f will ensure a higher order of approximation than $1/n$ to the Bernstein polynomials (with the exception of the linear function, for which the polynomial $B_n(f;\ x)$, when $n=1,\ 2,\ 3,\dots$ coincides with $f(x)$).

Consequently, the Bernstein polynomials have a disadvantage, determined by the fact that they don't react to the improvement of the smoothness of function they are generated from.

Thus the following question arises: Is it possible to correct the definition of the Bernstein polynomials, so that the new polynomials to react to the smoothness improvement of the generating function?

The answer happens to be positive.

Definition. A generalized Bernstein polynomial of (n, r)-th order for a function $f\in C^r[0,\ 1]$, $r=0,\ 1,\ 2,\dots$ is said to be the polynomial

$$B_{n,r}(f;\ x)=\sum_{k=0}^{n}\sum_{i=0}^{r}\frac{f^{(i)}(k/n)}{i!}(x-k/n)^i\binom{n}{k}x^k(1-x)^{n-k}. \tag{4}$$

For $r=0$ from (4) and (1) it follows the equality

$$B_{n,0}(f;\ x)=B_n(f;\ x).$$

Thus, it is seen that (4) is in fact a generalization of (1).

Moreover, for the polynomials (4) the following two statements, which are natural generalizations of the classical theorems of Popoviciu and Voronovskaya, hold.

Theorem 1. *Let $f\in C^r[0,\ 1]$, $r=0,\ 1,\ 2,\dots$ and $B_{n,r}(f;\ x)$ be the generalized Bernstein polynomial of order $(n,\ r)$ for f. Then*

$$\|f(\cdot)-B_{n,r}(f;\ \cdot)\|_C=O(n^{-r/2}\,\omega(f^{(r)};\ n^{-1/2})), \tag{5}$$

where

$$\|g(\cdot)\|_C=\sup\{|g(x)|: x\in[0,\ 1]\}$$

for arbitrary $g\in C[0,\ 1]$.

Theorem 2. *Let $f\in C^{r+2}[0,1]$, $r=0,1,2,\dots$ and $B_{n,r}(f;x)$ be the generalized Bernstein polynomial of order (n,r) for the function f. Then for every $x\in[0,1]$ it holds the following asymptotical equality*

$$B_{n,r}(f;x)=f(x)+\frac{(-1)^r f^{(r+1)}(x)S_{r+1}(x)}{(r+1)!\,n^{r+1}} \tag{6}$$

$$+\frac{(-1)^r(r+1)f^{(r+2)}(x)S_{r+2}(x)}{(r+2)!\,n^{r+2}}+\frac{\rho_{n,r}}{n^{r/2+1}}$$

with

$$S_m(x)=\sum_{k=0}^{n}(k-nx)^m\binom{n}{k}x^k(1-x)^{n-k},\quad m=0,1,2,\dots \tag{7}$$

and

$$\lim_{n\to\infty}\rho_{n,r}=0. \tag{8}$$

Proof of Theorem 1. Let $r=1,2,3,\dots$ (for $r=0$ see (2)). Using, the modified Taylor's formula

$$f(x)=\sum_{i=0}^{r}\frac{f^{(i)}(k/n)}{i!}(x-k/n)^i$$

$$+\frac{(x-k/n)^r}{(r-1)!}\int_0^1(1-t)^{r-1}\left(f^{(r)}(k/n+t(x-k/n))-f^{(r)}(k/n)\right)dt$$

and the obvious identity

$$\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}=1, \tag{$*$}$$

from the definitions of the modulus of continuity and (4) of the generalized Bernstein polynomial, for every $x\in[0,1]$ we obtain:

$$|f(x)-B_{n,r}(f;x)|=\Big|\sum_{k=0}^{n}f(x)\binom{n}{k}x^k(1-x)^{n-k}$$

$$-\sum_{k=0}^{n}\sum_{i=0}^{r}\frac{f^{(i)}(k/n)}{i!}(x-k/n)^i\binom{n}{k}x^k(1-x)^{n-k}\Big|$$

$$=\Big|\sum_{k=0}^{n}\frac{(x-k/n)^r}{(r-1)!}\int_0^1(1-t)^{r-1}\left(f^{(r)}(k/n+t(x-k/n))-f^{(r)}(k/n)\right)dt\times\binom{n}{k}x^k(1-x)^{n-k}\Big|$$

$$\leqq\sum_{k=0}^{n}\frac{|x-k/n|^r}{(r-1)!}\int_0^1(1-t)^{r-1}\,\omega(f^{(r)};t\,|x-k/n|)\,dt\binom{n}{k}x^k(1-x)^{n-k}.$$

Since $\omega(f;\ ps)\leqq(p+1)\ \omega(f;\ s)$, $p>0$, then

$$\omega(f^{(r)};\ t|x-k/n|.n^{1/2}.n^{-1/2})\leqq(t\,|x-k/n|n^{1/2}+1)\,\omega(f^{(r)};\ n^{-1/2}).$$

Thus we have the estimate

$$|f(x)-B_{n,r}(f;\ x)|\leqq\omega(f^{(r)};\ n^{-1/2}).A,\ x\in[0,\ 1], \tag{9}$$

with

$$A=\frac{1}{(r+1)!}n^{-r-1/2}\sum_{k=0}^{n}|k-nx|^{r+1}\binom{n}{k}x^k(1-x)^{n-k} \tag{10}$$
$$+\frac{1}{r!}n^{-r}\sum_{k=0}^{n}|k-nx|^{r}\binom{n}{k}x^k(1-x)^{n-k}.$$

Using the Cauchy–Bunyakovskii inequality and (*), we get

$$\sum_{k=0}^{n}|k-nx|^{r+1}\binom{n}{k}x^k(1-x)^{n-k}\leqq(S_{2r+2}(x))^{1/2},\ x\in[0,\ 1], \tag{11}$$

and

$$\sum_{k=0}^{n}|k-nx|^{r}\binom{n}{k}x^k(1-x)^{n-k}\leqq(S_{2r}(x))^{1/2},\ x\in[0,\ 1]. \tag{12}$$

On the other hand, it is known (see e. g. [4], p. 248, formula (269)), that for every $x\in[0,\ 1]$

$$|S_m(x)|\leqq K(m)n^{[m/2]}, \tag{13}$$

where $K(m)$ is a constant, depending on m, but not depending on n, and $[m/2]$ is the integer part of $m/2$.

From (10)–(13) the estimate

$$A\leqq n^{-r/2}\,[\frac{1}{(r+1)!}\sqrt{K(2r+2)}+\frac{1}{r!}\sqrt{K(2r)}=K_0(r)\,n^{-r/2}$$

follows. This estimate together with (9) implies (5) and the proof of the theorem is completed.

Proof of Theorem 2. Since from the hypothesis of the theorem, we have $f\in C^{r+2}[0,\ 1]$, $r=0,1,2,\dots$ it holds $f^{(i)}\in C^{r-i+2}[0,\ 1]$ and for arbitrary y and x from the segment $[0,\ 1]$ it holds the Taylor formula with remainder in Peano's form

$$f^{(i)}(x)=\sum_{s=0}^{r-i+2}\frac{f^{(i+s)}(x)}{s!}(t-x)^s+\alpha_i(i-x)\,(t-x)^{r-i+2}, \tag{14}$$

where

(15) $$\lim_{q\to\infty} \alpha_i(q)=0, \ \alpha_i(0)=0,$$

and $\alpha_i(q) \cdot q^{r-i+2}$ is a continuous function on [0, 1]. Then substituting (14) (with $t=k/n$) in (4) and taking into account (7), we obtain the equality

(16) $$B_{n,r}(f;\ x)=J+R_{n,r}(x),$$

where

(17) $$J=\sum_{i=0}^{r}\sum_{p=i}^{r+2}\frac{f^{(p)}(x)}{i!(p-i)!}(-1)^i n^{-p} S_p(x)$$

and

(18) $$R_{n,r}(x)=\sum_{k=0}^{n}\sum_{i=0}^{r+2}\frac{1}{i!}\alpha_i(k/n-x)(-1)^i (k/n-x)^{r+2}\binom{n}{k}x^k(1-x)^{n-k}.$$

We will make some simple transformations of the expression (17):

$$J=f(x)+\sum_{p=i}^{r}\frac{f^{(p)}(x)}{p!n^p}S_p(x)\sum_{k=0}^{p}\binom{p}{k}(-1)^k$$
$$+\frac{f^{(r+1)}(x)}{(r+1)!n^{r+1}}S_{r+1}(x).\sum_{k=0}^{r}\binom{r+1}{k}(-1)^k$$
$$+\frac{f^{(r+2)}(x)}{(r+2)!n^{r+2}}S_{r+2}(x).\sum_{k=0}^{r}\binom{r+2}{k}(-1)^k.$$

Using the elementary identities

$$\sum_{k=0}^{p}\binom{p}{k}(-1)^k=(1-1)^p=0,\ p=1, 2,\dots,r,$$

$$\sum_{k=0}^{r}\binom{r+1}{k}(-1)^k=(-1)^r$$

and

$$\sum_{k=0}^{r}\binom{r+2}{k}(-1)^k=(-1)^r.(r+1),$$

we receive the representation

$$J=f(x)+(-1)^r\frac{f^{(r+1)}(x)\,S_{r+1}(x)}{(r+1)!\,n^{r+1}}+(-1)^r\frac{(r+1)f^{(r+2)}(x)\,S_{r+2}(x)}{(r+2)!\,n^{r+2}},$$

which together with (16) gives (6).

We should also prove (8) in order to complete the proof of the theorem. Let $\varepsilon>0$ be arbitrary. Since

$$\lim_{q\to 0} \sum_{i=0}^{r} \frac{1}{i!}|\alpha_i(q)|=0,$$

then there exists a large integer n, such that for every $q\in[0, 1]$, with $|q|<n^{-1/4}$, it holds the inequality

$$\sum_{i=0}^{r} \frac{1}{i!}|\alpha_i(q)| < \frac{\varepsilon}{2\sqrt{K(2r+4)}}, \tag{19}$$

where $K(m)$ is the constant from (13).

With this choice of the number n, we split the set $\{0, 1, \ldots, n\}$ into two subsets $\Delta(n)$ and $\Gamma(n)$, in such a way that in $\Delta(n)$ are included those k, for which $|k/n-x|<n^{-1/4}$ and in $\Gamma(n)$ those, for which $|k/n-x|\geqq n^{-1/4}$. Then from (18) it follows the estimate

$$|R_{n,r}(x)|\leqq \frac{\varepsilon}{2\sqrt{K(2r+4)}} \sum_{k\in\Delta(n)} \left|\frac{k}{n}-x\right|^{r+2}\binom{n}{k}x^k(1-x)^{n-k}+B(x),\ 0\leqq x\leqq 1,$$

where

$$B(x)= \sum_{k\in\Gamma(n)} \sum_{i=0}^{r} \frac{1}{i!}\left|\alpha_i\left(\frac{k}{n}-x\right)\left(\frac{k}{n}-x\right)^{r-i+2}\right| . \left|\frac{k}{n}-x\right|^{i}\binom{n}{k}x^k(1-x)^{n-k}. \tag{20}$$

Using the Cauchy—Bunyakovskii inequality, the elementary identity (*) and the inequality (13), we get

$$\frac{\varepsilon}{2\sqrt{K(2r+4)}}. \sum_{k\in\Delta(n)} \left|\frac{k}{n}-x\right|^{r+2}\binom{n}{k}x^k(1-x)^{n-k}$$

$$\leqq \frac{\varepsilon}{2n^{r+2}\sqrt{K(2r+4)}}. \sum_{k=0} |k-nx|^{r+2}\binom{n}{k}x^k(1-x)^{n-k}$$

$$\leqq \frac{\varepsilon}{2.n^{r+2}\sqrt{K(2r+4)}}.\sqrt{S_{2r+4}(x)}.\sqrt{S_0(x)}$$

$$\leqq \frac{\varepsilon}{2.n^{r+2}\sqrt{K(2r+4)}}.\sqrt{K(2r+4).n^{r+2}} = \frac{\varepsilon}{2.n^{r/2+1}}$$

and, hence

$$|R_{n,r}(x)| < \frac{\varepsilon}{2.n^{r/2+1}}+B(x),\ 0\leqq x\leqq 1. \tag{21}$$

Since the function

$$\sum_{i=0}^{r} \frac{1}{i!}|\alpha_i(q).q^{r-i+2}|,\ q\in[0, 1],$$

is bounded, then there exists a constant $M(r)>0$, such that for every $q\in[0, 1]$ the inequality

$$\sum_{i=0}^{r}\frac{1}{i!}|\alpha_i(q)\,.\,q^{r-i+2}|\leqq M(r)$$

holds. Hence, using (20), the inequality

$$\sum_{k\in\Gamma(n)}\binom{n}{k}x^k(1-x)^{n-k}\leqq\frac{K(2r+6)}{n^{r/2+3/2}}$$

(see e. g. [4], p. 248, Lemma 3) and

$$|k/n-x|^i\leqq 1,\ i=0,\ 1,\ 2,\dots,$$

we get

$$|B(x)|\leqq\sum_{k\in\Gamma(n)}\sum_{r=0}^{r}\frac{1}{i!}|\alpha_i(\frac{k}{n}-x)(\frac{k}{n}-x)^{r-i+2}|.\binom{n}{k}x^k(1-x)^{n-k}$$

$$\leqq M(r)\sum_{k\in\Gamma(n)}\binom{n}{k}x^k(1-x)^{n-k}\leqq M(r)\frac{K(2r+6)}{n^{r/2+3/2}}.$$

If it is necessary, we increase n until we obtain

$$M(r)\,.\,K(2r+6)\,.\,n^{-1/2}<\frac{\varepsilon}{2}$$

and

$$|B(x)|<\frac{\varepsilon}{2\,.\,n^{r/2+1}},\qquad\forall\,x\in[0,\ 1].$$

Hence, using the estimate for $B(x)$ and (21), we get

$$|\rho_{n,r}(x)|:=n^{r/2+1}\,.\,|R_{n,r}(x)|\leqq n^{r/2+1}\,.\,\|R_{n,r}(\cdot)\|_C<\varepsilon$$

and thus the proof is completed.

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