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On Generalized Fractional q-Integrals

Mumtaz Ahmad Khan

Presented by P. Kenderov

The present paper deals with two generalized fractional q-integrals unifying the known fractional q-integral operators due to W. A. Al-Salam [2], R. P. Agarwal [1]., M. Upadhyay [7], M. A. Khan [6] and W. A. Al-Salam and A. Verma [3].

1. Introduction

In 1951, A. Erdelyi [4] defined the operators of fractional integration:

$$If=I[f(x);\ m,\ \alpha,\ \eta] \tag{1.1}$$

$$=\frac{m}{\Gamma(\alpha)}x^{-\eta-m\alpha+m-1}\int_0^x (x^m-u^m)^{\alpha-1}u^\eta f(u)\,du$$

and

$$Kf=K[f(x);\ m,\ \alpha,\ \eta] \tag{1.2}$$

$$=\frac{m}{\Gamma(\alpha)}x^\eta\int_x^\infty (u^m-x^m)^{\alpha-1}u^{-\eta-m\alpha+m-1}f(u)\,du,$$

where $\alpha>0$, $m>0$.

In 1974, the present author [6] defined q-analogues of the above operators in the following form:

$$I_{m,q}^{\eta,\alpha}f(x)=\frac{mx^{-\eta-m\alpha+m-1}}{\Gamma_q(\alpha)}\int_0^x (x^m-t^mq^m)_{\alpha-1}t^\eta f(t)\,d(t;q), \tag{1.3}$$

$$K_{m,q}^{\eta,\alpha}f(x)=\frac{mq^{-\eta}x^\eta}{\Gamma_q(\alpha)}\int_x^\infty (t^m-x^m)_{\alpha-1}t^{-\eta-m\alpha+m-1}f(tq^{1-\alpha})\,d(t;q) \tag{1.4}$$

where $\alpha\neq 0,\ -1,\ -2,\dots$

For $m=1$ (1.3) reduces to the following fractional q-integral operator due to W. A. Agarwal [1]:

(1.5) $$I_q^{\eta,\alpha} f(x) = \frac{x^{-\eta-\alpha x}}{\Gamma_q(\alpha)} \int_0 (x-tq)_{\alpha-1} t^{\eta} f(t)\, \mathrm{d}(t;q),$$

while (1.4) reduces to the fractional q-integral operator

(1.6) $$K_q^{\eta,\alpha} f(x) = \frac{q^{-\eta} x^{\eta}}{\Gamma_q(\alpha)} \int_x^{\infty} (t-x)_{\alpha-1} t^{-\eta-\alpha} f(tq^{1-\alpha})\, \mathrm{d}(t;q)$$

which is due to W. A. Al-Salam [2].

In 1975, W. A. Al-Salam and A. Verma [3] studied the following operators:

(1.7) $$_qI_{x\lambda}^{\alpha}\{f(t)\} = \frac{[\lambda]}{G_h(\alpha)} \int_0^x [x^{\lambda} - q^{\lambda} t^{\lambda}]_{\alpha-1,h} t^{\lambda-1} f(t)\, \mathrm{d}(t;q)$$
$$= (1-h)^{\alpha} x^{\lambda\alpha} \sum_{j=0}^{\infty} h^j \frac{(h^{\alpha})_{j,h}}{(h)_{j,h}} f(xq^j)$$

and

(1.8) $$_qI_{x\lambda}^{\eta,\alpha}\{f(t)\} = \frac{(1-h)x^{-\eta\lambda-\lambda\alpha}}{(1-q)G_h(\alpha)} \int_0^x [x^{\lambda} - q^{\lambda} t^{\lambda}]_{\alpha-1,h} t^{\eta\lambda+\lambda-1} f(t)\, \mathrm{d}(t;q),$$

where $h = q^{\lambda}$ and $G_q(\alpha) = \Gamma_q(\alpha)$.

These operators differ from those defined in (1.3) and (1.4), since the products in the integrands of these operators advance in powers of q^{λ} unlike in powers of q in the operators (1.3) and (1.4).

It may be noted that inspite of the fact that the operator (1.5) is a particular case of all three operators $I_q[(a); (b); z, \eta : f(x)]$ of M. Upadhyay [7], $_qI_{x\lambda}^{\eta,\alpha}\{f(t)\}$ of W. A. Al-Salam and A. Verma [3] and $I_{m,q}^{\eta,\alpha} f(x)$ of M. A. Khan [6], yet all these are three extensions of (1.5) unconnected with each other. Similar remarks are applied to the operators $K_q[(a); (b); z, \eta : f(x)]$ of M. Upadhyay [7] and $K_{m,q}^{\eta,\alpha} f(x)$ of M. A. Khan [6] which contain the operator (1.6) as a common particular case. This led the author to unify the known fractional q-integral operators. The present paper deals with a study of two such unified operators. The results obtained here generalize those obtained by M. Upadhyay [7] and by the present author [6].

The following definitions and notations will be used further in this paper:

(1.9) $$[\alpha] = (1-q^{\alpha})/(1-q),$$

(1.10) $$(q^{\alpha})_n = (1-q^{\alpha})(1-q^{\alpha+1})\dots(1-q^{\alpha+n-1});\ (q^{\alpha})_0 = 1,$$

(1.11) $$_A\Phi_B^{(q)}\left[\begin{matrix}(a); & x\\ (b); & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(q^{a_1})_n (q^{a_2})_n \dots (q^{a_A})_n x^n}{(q)_n (q^{b_1})_n (q^{b_2})_n \dots (q^{b_B})_n},\ |x| < 1,$$

(1.12) $$\Gamma_q(\alpha) = \frac{(1-q)_{\alpha-1}}{(1-q)^{\alpha-1}},\ (\alpha \neq 0, -1, -2, \dots),$$

(1.13) $$\int_0^x f(t)\,d(t;q)=x(1-q)\sum_{n=0}^{\infty} q^n f(xq^n).$$

(1.14) $$\int_x^{\infty} f(t)\,d(t;q)=x(1-q)\sum_{n=1}^{\infty} q^{-n} f(xq^{-n}).$$

(1.15) $$\int_0^{\infty} f(t)\,d(t;q)=(1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n).$$

We shall also use the following Heine's theorem:

(1.16) $${}_1\Phi_0[q^a;\ -;\ z]=\frac{(1-q^a z)_\infty}{(1-z)_\infty}=\frac{1}{(1-z)_a}.$$

2. Unifying operators

We now introduce the following generalized fractional q-integral operators:

(2.1) $$I_q[(a);\ (b);\ \omega,\lambda;\ z,\mu;\ \eta:f(x)]$$

$$=\frac{x^{-\eta\lambda-\lambda}}{(1-q)}\int_0^x t^{\eta\lambda+\lambda-1}\,{}_A\Phi_B^{(q^\lambda)}\left[\begin{matrix}(a); & \omega^\lambda z^\mu t^\mu/x^\mu\\ (b); & \end{matrix}\right]f(t)\,d(t;q)$$

$$=\sum_{k=0}^{\infty} q^{k(\eta+1)\lambda}\,{}_A\Phi_B^{(q^\lambda)}\left[\begin{matrix}(a); & \omega^\lambda z^\mu q^{k\mu}\\ (b); & \end{matrix}\right]f(xq^k).$$

(2.2) $$K_q[(a);\ (b);\ \omega,\lambda;\ z,\mu;\ \eta:f(x)]$$

$$=\frac{x^{\eta\lambda+\lambda-1}q^{-\eta\lambda-\lambda+1}}{(1-q)}\int_x^{\infty} t^{-\eta\lambda-\lambda}\,{}_A\Phi_B^{(q^\lambda)}\left[\begin{matrix}(a); & \omega^\lambda z^\mu x^\mu/t^\mu\\ (b); & \end{matrix}\right]f(t)\,d(t;q)$$

$$=\sum_{k=0}^{\infty} q^{k(\eta\lambda+\lambda-1)}\,{}_A\Phi_B^{(q^\lambda)}\left[\begin{matrix}(a); & \omega^\lambda z^\mu q^{k\mu+\mu}\\ (b); & \end{matrix}\right]f(xq^{-k-1}).$$

Particular cases (i): For $\lambda=\mu=\omega=1$, (2.1-2) reduce to the following operators due to M. Upadhyay [7]:

(2.3) $$I_q[(a);\ (b);\ z,\eta:f(x)]$$

$$=\frac{x^{-\eta-1}}{(1-q)}\int_0^x t^\eta\,{}_A\Phi_B^{(q)}[(a);\ (b);\ zt/x]f(t)\,d(t;q)$$

$$=\sum_{k=0}^{\infty} q^{k(\eta+1)}\,{}_A\Phi_B^{(q)}[(a);\ (b);\ zq^k]f(xq^k)$$

and

(2.4) $$K_q[(a);\ (b);\ z,\ \eta : f(x)]$$

$$= \frac{x^{\eta} q^{-\eta}}{(1-q)} \int_x^{\infty} t^{-1-\eta}\, {}_A\Phi_B^{(q)}[(a);\ (b);\ zx/t] f(t)\, d(t;q)$$

$$= \sum_{k=0}^{\infty} q^{k\eta}\, {}_A\Phi_B^{(q)}[(a);\ (b);\ zq^{k+1}] f(xq^{-k-1}).$$

(ii) For $\lambda=1$, $\mu=m$, $\omega=q^{\alpha-1}$, $B=0$, $A=1$, $a_1=-\alpha+1$ and $z=q$, we get

$$\frac{m(1-q)}{\Gamma_q(\alpha)} I_q[-\alpha+1;\ -;q^{\alpha-1},\ 1;\ q,\ m,\ \eta;\ f(x)]$$

$$= \frac{mx^{-\eta-1}}{\Gamma_q(\alpha)} \int_0^x t^{\eta}\, {}_1\Phi_0\left[\begin{matrix} q^{-\alpha+1}; & q^{m+\alpha-1}\, t^m/x^m \\ \text{---} & ; \end{matrix}\right] f(t)\, d(t;q)$$

$$= \frac{mx^{-\eta-m\alpha+m-1}}{\Gamma_q(\alpha)} \int_0^x t^{\eta}(x^m - q^m t^m)_{\alpha-1} f(t)\, d(t;q)$$

$$= I_{m,q}^{\eta,\alpha} f(x)$$

which is (1.3).

(iii) For $\lambda=\mu$, $\omega=1$, $B=0$, $A=1$, $a=-\alpha+1$, $z=q^{\alpha}$, we get by setting $h=q^{\lambda}$,

$$\frac{(1-h)}{G_h(\alpha)} I_q[-\alpha+1;\ \text{---}\ ;\ 1,\ \lambda;\ q^{\alpha},\ \lambda;\ \eta : f(x)]$$

$$= \frac{x^{-\eta\lambda-\lambda}(1-h)}{(1-q)G_h(\alpha)} \int_0^x t^{\eta\lambda+\lambda-1}\, {}_1\Phi_0\left[\begin{matrix} h^{-\alpha+1}; & q^{\lambda\alpha}\, t^{\lambda}/x^{\lambda} \\ \text{---} & ; \end{matrix}\right] f(t)\, d(t;q)$$

$$= \frac{(1-h)x^{-\eta\lambda-\lambda}}{(1-q)G_h(\alpha)} \int_0^x t^{\eta\lambda+\lambda-1}\, {}_1\Phi_0\left[\begin{matrix} h^{-\alpha+1}; & h^{\alpha-1}\, q^{\lambda}\, t^{\lambda}/x^{\lambda} \\ \text{---} & ; \end{matrix}\right] f(t)\, d(t;q)$$

$$= \frac{(1-h)x^{-\eta\lambda-\lambda}}{(1-q)G_h(\alpha)} \int_0^x t^{\eta\lambda+\lambda-1}(x^{\lambda} - q^{\lambda} t^{\lambda})_{\alpha-1,h} f(t)\, d(t;q)$$

$$= {}_qI_{x,\lambda}^{\eta,\alpha}\{f(t)\},$$

which is (1.8)

(iv) For $B=0$, $A=1$, $a_1=-\alpha+1$, $\lambda=1$, $\mu=m$, $\omega=q^{\alpha-1}$, $z=1$ and $f(x)$ replaced by $f(xq^{1-\alpha})$, (2.2) reduces to (1.4).

A study of these fractional q-integral operators is expected to be useful in the development of the q-function theory, playing an important role in combinatory analysis.

3. Some elementary properties

The following formal properties of the generalized fractional q-integral operators (2.1) and (2.2) can be easily obtained:

(3.1) $$x^{\lambda c} I_q[(a);\ (b);\ \lambda, \omega;\ z, \mu;\ \eta : f(x)]$$
$$= I_q[(a);\ (b);\ \lambda, \omega;\ z, \mu;\ \eta - c : x^{\lambda c} f(x)],$$

(3.2) $$x^{\lambda c} K_q[(a);\ (b);\ \lambda, \omega;\ z, \mu;\ \eta : f(x)]$$
$$= K_q[(a);\ (b);\ \lambda, \omega;\ z, \mu;\ \eta + c : (xq)^{\lambda c} f(x)],$$

(3.3) $$\text{if } I_q f(x) = g(x), \text{ then } I_q f(\lambda x) = g(\lambda x),$$

(3.4) $$\text{if } K_q f(x) = g(x), \text{ then } K_q f(\lambda x) = g(\lambda x).$$

The last two equations express a homogeneity of the operators. They show that given a function $f(xy)$ there is no difference whether the operators are applied with respect to x, y or to $\omega = xy$.

4. q-Mellin transforms of (2.1) and (2.2)

Theorem 1: *If* $\sum_{r=-\infty}^{\infty} |q^{rs} f(q^r)|$ *converges*, $|q|<1$,

$\mathrm{Re}(\mu) > 0$, $|\omega^\lambda z^\mu| < 1$ and $\mathrm{Re}(\eta\lambda + \lambda - s) > 0$, then

(4.1) $$M_q\{I_q[(a);\ (b);\ \lambda, \omega;\ z, \mu;\ \eta : f(x)]\}$$
$$= \{1 - q^{\eta\lambda + \lambda - s}\}^{-1}\ {}_{A+1}\Phi_{B+1}\left[\begin{matrix} h^{(a)} : q^{\eta\lambda+\lambda-s};\ \omega^\lambda z^\mu \\ h^{(b)} : q^{\eta\lambda+\lambda+\mu-s}; \end{matrix}\right] M_q\{f(x)\}$$

where both in the numerator and the denominator of the "bibasic" series on the right, the terms before the colon are on the base $h = q^\lambda$ and those after it are on the base q^μ and the q-analogue of Mellin transform of $f(x)$ is defined as

$$M_q f(x) = \int_0^\infty x^{s-1} f(x)\, d(x;\ q).$$

Proof: $$M_q\{I_q[(a);\ (b);\ \lambda, \omega;\ z, \mu;\ \eta : f(x)]$$
$$= \frac{1}{(1-q)} \int_0^\infty x^{s-1-\eta\lambda-\lambda} \{ \int_0^x t^{\eta\lambda+\lambda-1}\ {}_A\Phi_B^{(h)}\left[\begin{matrix} (a);\ \omega^\lambda z^\mu t^\mu / x^\mu \\ (b); \end{matrix}\right] f(t)\, d(t;q)\}\, d(x;q)$$
$$= \int_0^\infty x^{s-\eta\lambda-\lambda} [\sum_{k=0}^\infty q^k (xq^k)^{\eta\lambda+\lambda-1}\ {}_A\Phi_B^{(h)}\left[\begin{matrix} (a);\ \omega^\lambda z^\mu q^{k\mu} \\ (b); \end{matrix}\right] f(xq^k)]\, d(x;q)$$

$$=(1-q)\sum_{r=-\infty}^{\infty} q^{rs}\sum_{k=0}^{\infty} q^{k\lambda(\eta+1)}{}_A\Phi_B^{(h)}\left[\begin{matrix}(a); & \omega^\lambda z^\mu q^{k\mu}\\ (b); & \end{matrix}\right]f(q^{r+k})$$

$$=(1-q)\sum_{n=-\infty}^{\infty} q^{ns}f(q^n)\sum_{k=0}^{\infty} q^{k(\eta\lambda+\lambda-s)}{}_A\Phi_B^{(h)}\left[\begin{matrix}(a); & \omega^\lambda z^\mu q^{k\mu}\\ (b); & \end{matrix}\right]$$

$$=(1-q)\sum_{k=0}^{\infty} q^{k(\eta\lambda+\lambda-s)}\sum_{j=0}^{\infty}\frac{[h^{(a)}]_j\omega^{\lambda j}z^{\mu j}q^{k\mu j}}{(h)_j[h^{(b)}]_j}\sum_{n=-\infty}^{\infty} q^{ns}f(q^n)$$

$$=\sum_{j=0}^{\infty}\frac{[h^{(a)}]_j\omega^{\lambda j}z^{\mu j}}{(h)_j[h^{(b)}]_j}\sum_{k=0}^{\infty} q^{k(\eta\lambda+\lambda-s+\mu j)}\int_x^{\infty} x^{s-1}f(x)\,d(x;\,q)$$

$$=\sum_{j=0}^{\infty}\frac{[h^{(a)}]_j\omega^{\lambda j}z^{\mu j}}{(h)_j[h^{(b)}]_j(1-q)^{\eta\lambda+\lambda-s+\mu j)}}M_q\{f(x)\}$$

$$=\frac{1}{(1-q^{\eta\lambda+\lambda-s})}\sum_{j=0}^{\infty}\frac{[h^{(a)}]_j\omega^{\lambda j}z^{\mu j}(q^{\eta\lambda+\lambda-s})_{j,q^\mu}}{(h)_j[h^{(b)}]_j(q^{\eta\lambda+\lambda+\mu-s})_{j,q^\mu}}M_q\{f(x)\}$$

$$=\{1-q^{\eta\lambda+\lambda-s}\}^{-1}{}_{A+1}\Phi_{B+1}\left[\begin{matrix}h^{(a)}:q^{\eta\lambda+\lambda-s}; & \omega^\lambda z^\mu\\ h^{(b)}:q^{\eta\lambda+\lambda+\mu-s}; & \end{matrix}\right]M_q\{f(x)\}.$$

This proves the theorem.

Proceeding as above, we have

Theorem 2. *If* $\sum_{r=-\infty}^{\infty}|q^{rs}f(q^r)|$ *converges,* $|q|<1$, $\mathrm{Re}(\mu)>0$, $|\omega^\lambda z^\mu|<1$ *and* $\mathrm{Re}(\eta\lambda+\lambda+s)>1$, *then*

(4.2) $$M_q\{K_q[(a);\ (b);\ \lambda,\omega;\ z,\mu;\ \eta:f(x)]$$

$$=q^s\{1-q^{\eta\lambda+\lambda+s-1}\}^{-1}{}_{A+1}\Phi_{B+1}\left[\begin{matrix}h^{(a)}:q^{\eta\lambda+\lambda+s-1}; & \omega^\lambda z^\mu q^\mu\\ h^{(b)}:q^{\eta\lambda+\lambda+s+\mu-1}; & \end{matrix}\right]M_q\{f(x)\},$$

where in the numerator and the denominator of the "bibasic" series on the right, the terms before the colon are on the base $h=q^\lambda$ *and those after it are on the base* q^μ.

Particular cases of Theorem 1 and 2. We now consider certain particular cases of Theorem 1 and 2 (in the form of Corollaries): (i) Setting $\lambda=\mu=\omega=1$ in Theorems 1 and 2, we have

Corollary 1. *If* $\sum_{r=-\infty}^{\infty}|q^{rs}f(q^r)|$ *converges,* $|q|<1$, $|z|<1$ *and* $\mathrm{Re}(\eta-s)>-1$ *then*

$$(4.3)\quad M_q\{I_q[(a);\ (b);\ z,\eta:f(x)]\}=(1-q^{\eta+1-s})^{-1}\,{}_{A+1}\Phi_{B+1}^{(q)}\left[\begin{matrix}(a),\ \eta+1-s;\ z\\ (b),\ \eta+2-s;\end{matrix}\right]\times M_q\{f(x)\},$$

and

Corollary 2. *If* $\sum_{r=-\infty}^{\infty}|q^{rs}f(q^r)|$ *converges,* $|q|<1$, $|z|<1$ *and* $\mathrm{Re}(\eta+s)>0$ *then*

$$(4.4)\quad M_q\{K_q[(a);\ (b);\ z,\eta:f(x)]\}=q^s(1-q^{\eta+s-1})_{A+1}\Phi_{B+1}\left[\begin{matrix}(a),\ \eta+s;\ zq\\ (b),\ \eta+s+1;\end{matrix}\right].\times M_q\{f(x)\}.$$

(ii) For $A=1$, $a_1=-\alpha+1$, $B=0$, $\lambda=1$, $\mu=m$, $z=1$, $\omega=q^{\alpha-1}$ and $f(x)$ replaced by $f(xq^{1-\alpha})$ in Theorem 2, we get

Corollary 3. *If* $\sum_{r=-\infty}^{\infty}|q^{rs}f(q^r)|$ *is convergent,* $|q|<1$, m *is a positive integer,* $\mathrm{Re}(\eta-s)=-1$ *and* $\mathrm{Re}(\alpha)>1$ *then*

$$(4.5)\quad M_qI_{m,q}^{\eta,\alpha}f(x)=\frac{m(1-q)}{(1-q^{1+\eta-s})\Gamma_q(\alpha)}\Phi\left[\begin{matrix}q^{1-\alpha}:q^{1+\eta-s};\ q^{m+\alpha-1}\\ q:q^{1+\eta-s+m};\end{matrix}\right]M_q\{f(x)\}$$

where in the numerator and the denominator of the "bibasic" series, the terms before the colon are on the base q and those after it are on the base q^m.

Similarly, from Theorem 2, we have

Corollary 4. *If* $\sum_{r=-\infty}^{\infty}|q^{rs}f(q^{r-s})|$ *is convergent,* $|q|<1$, m *is a positive integer,* $\mathrm{Re}(\eta+s)>0$ *and* $\mathrm{Re}(\alpha)>1$, *then*

$$(4.6)\quad M_qK_{m,q}^{\eta,\alpha}f(x)=\frac{m(1-q)}{(1-q^{\eta+s})\Gamma_q(\alpha)}\Phi\left[\begin{matrix}q^{1-\alpha}:q^{\eta+s};\ q^{m+\alpha-1}\\ q:q^{\eta+s+m};\end{matrix}\right]M_q\{f(xq^{-\alpha})\},$$

where in the numerator and the denominator of the "bibasic" series, the terms before the colon are on the base q and those after it are on the base q^m.

Results (4.3-4) are due to M. Upadhyay [7], while (4.5-6) are due to the author [6].

5. Fractional integration by parts

Theorem 3. *If* $\sum_{r=-\infty}^{\infty}|q^{r(\eta\lambda+\lambda)}f(q^r)|$ *and* $\sum_{r=-\infty}^{\infty}|q^{-r(\eta\lambda+\lambda-1)}g(q^r)|$ *are convergent,* $|q|<1$, $\mathrm{Re}(\mu)>0$, $|\omega^\lambda z^\mu|<1$ and $\mathrm{Re}(\eta\lambda+\lambda)>0$, *then*

(5.1) $$\int_0^\infty f(x)\, K_q[(a);\ (b);\ \lambda, \omega;\ \mu;\ \eta : g(x)]\, d(x;\ q)$$

$$= \int_0^\infty g(xq^{-1})\, I_q[(a);\ (b);\ \lambda, \omega;\ zq, \mu;\ n : f(x)]\, d(x;\ q)$$

Proof.

$$\int_0^\infty f(x)\, K_q[(a);\ (b);\ \lambda, \omega;\ z, \mu;\ \eta : g(x)]\, d(x;\ q)$$

$$=(1-q) \sum_{r=-\infty}^{\infty} q^r f(q^r) \sum_{k=0}^{\infty} q^{k(\eta\lambda+\lambda-1)}\, {}_A\Phi_B^{(h)}\left[\begin{matrix}(a); & \omega^\lambda z^\mu q^{k\mu+\mu} \\ (b); & \end{matrix}\right] g(q^{r-k-1})$$

$$=(1-q) \sum_{n=-\infty}^{\infty} q^n g(q^{n-1}) \sum_{k=0}^{\infty} q^{k(\eta\lambda+\lambda)}\, {}_A\Phi_B^{(h)}\left[\begin{matrix}(a); & \omega^\lambda z^\mu q^{k\mu+\mu} \\ (b); & \end{matrix}\right] f(q^{n+k})$$

$$= \int_0^\infty g(xq^{-1})\, I_q[(a);\ (b);\ \lambda, \omega;\ zq, \mu;\ \eta : f(x)]\, d(x;\ q)$$

which proves the theorem.

Particular cases of Theorem 3.

(i) For $\lambda=\omega=\mu=1$, (51.) reduces to the following:

Corollary 5. *If* $\sum_{r=-\infty}^{\infty} |q^{r(\eta+1)} f(q^r)|$ *and* $\sum_{r=-\infty}^{\infty} |q^{-r\eta} g(q^r)|$ *are convergent,* $|q|<1$, $|z|<1$ *and* $\mathrm{Re}(\eta)>-1$, *then*

(5.2) $$\int_0^\infty f(x)\, K_q[(a);\ (b);\ z, \eta : g(x)]\, d(x, q)$$

$$= \int_0^\infty g(xq^{-1})\, I_q[(a);\ (b);\ zq, \eta : f(x)]\, d(x;\ q)$$

(ii) For $\lambda=1$, $\mu=m$, $z=1$, $\omega=q^{\alpha-1}$, $B=0$, $A=1$, $a_1=-\alpha+1$ and $g(x)$ replaced by $g(xq^{1-\alpha})$, (5.1) yields

Corollary 6. *If* $\sum_{r=-\infty}^{\infty} |q^{r(\eta+1)} f(q^r)|$ *and* $\sum_{r=-\infty}^{\infty} |q^{-r\eta} g(q^{r-\alpha})|$ *are convergent,* $|q|<1$, *m is a positive integer,* $\mathrm{Re}(\eta)>-1$ *and* $Re(\alpha)>1$, *then*

(5.3) $$\int_0^\infty f(x)\, K_{m,q}^{\eta;\alpha} g(x)\, d(x;\ q) = \int_0^\infty g(xq^{-\alpha})\, I_{m,q}^{\eta;\alpha} f(x)\, d(x;\ q).$$

(iii) For $\lambda=\mu=z=1$, $\omega=q^{\alpha-1}$, $B=0$, $A=1$, $a_1=-\alpha+1$, (5.1) becomes

Corollary 7. *Let* $\sum_{r=-\infty}^{\infty} |q^{r(\eta+1)} f(q^r)|$ *and* $\sum_{r=-\infty}^{\infty} |q^{-r\eta} g(q^r)|$ *are convergent,* $|q|<1$, $\operatorname{Re}(\alpha)>1$, $\operatorname{Re}(\eta)>-1$ *then*

$$(5.4)\qquad \int_0^{\infty} f(x)\, K_q^{\eta,\alpha} g(x)\, \mathrm{d}(x;\, q) = \int_0^{\infty} g(xq^{-\alpha})\, I_q^{\eta,\alpha} f(x)\, \mathrm{d}(x;\, q).$$

The result (5.2) is due to M. Upadhyay [7], (5.3) is due to the author [6] and (5.4) is due to R. P. Agarwal [1].

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Dept. of Applied Mathematics
Faculty of Engineering
Aligarh Muslim University,
Aligarh-202002, U.P.,
INDIA

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