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On Domain Decomposition Preconditioning of Plane Elasticity Problems*

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We have studied an algebraic formulation of the domain decomposition method for solving 2D elasticity boundary value problems. The goal of the investigation is the constructing of robust vectorizable algorithms for solving the related large scale finite element linear systems.

The preconditioning matrix is defined under the assumption that the domain Ω is an union of macroelements (subdomains). The obtained results are based on the Korn's inequality and on a local analysis of the relative condition number of the proposed preconditioner.

Numerical tests demonstrating the convergence of the corresponding preconditioned conjugate gradient method are presented.

1. Introduction

The analysis of the deformation of the elastic solids leads to solution of a coupled system of differential equations. We assume that the material is homogeneous, isotropic and linear elastic, and also the displacements are small. The 2D problem of the theory of the elasticity can be formulated in the terms of the displacements $\mathbf{u}^T=(u, v)$. If the strains $\varepsilon^t=(\varepsilon_1, \varepsilon_2, \varepsilon_{12})$ and the stresses $\sigma^t=(\sigma_1, \sigma_2, \sigma_{12})$ are defined as follows

$$\varepsilon_1=\frac{\partial u}{\partial x}, \quad \varepsilon_2=\frac{\partial v}{\partial y}, \quad \varepsilon_{12}=\frac{1}{2}\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]$$

$$\sigma_1=(\lambda+2\mu)\varepsilon_1+\lambda\varepsilon_2, \quad \sigma_2=\lambda\varepsilon_1+(\lambda+2\mu)\varepsilon_2, \quad \sigma_{12}=2\mu\varepsilon_{12},$$

then the equilibrium equations takes the form

$$(1.1)\qquad \begin{aligned} -\left[\frac{\partial}{\partial x}\sigma_1+\frac{\partial}{\partial y}\sigma_{12}\right]&=f_1\\ -\left[\frac{\partial}{\partial x}\sigma_{12}+\frac{\partial}{\partial y}\sigma_2\right]&=f_2 \end{aligned} \qquad (x, y)\in\Omega$$

where $\mathbf{f}^t=(f_1,f_2)$ is the vector of the body forces. We consider the model problem with homogeneous Dirichlet boundary conditions so that

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(1.2) $$u=0, \quad v=0, \quad (x, y)\in\Gamma=\partial\Omega.$$

Usually one introduces Young's modulus E and Poisson's ratio ν. The material constants λ and μ have the following presentation

$$\lambda=\nu E/[(1+\nu)(1-2\nu)], \quad \mu=E/[2(1+\nu)],$$

where $E>0$ and $\nu\in(0, 1/2)$. After the substitution $\tilde{\nu}=\nu/(1-\nu)$ we obtain the system

$$-\left[\frac{\partial^2 u}{\partial x^2}+\frac{1-\tilde{\nu}}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\tilde{\nu}}{2}\frac{\partial^2 v}{\partial x\partial y}\right]=f_1^*$$

(1.3) $$(x, y)\in\Omega$$

$$-\left[\frac{1-\tilde{\nu}}{2}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{1+\tilde{\nu}}{2}\frac{\partial^2 u}{\partial x\partial y}\right]=f_2^*$$

Here $f_i^*=f_i(1-\tilde{\nu})(1+\nu)/E$, $i=1, 2$, and $\tilde{\nu}\in(0, 1)$. Let us define the spaces

(1.4) $$V=V_1\times V_2, \quad V_i=\{u\in H_i(\Omega) : u|_\Gamma=0\}, \quad i=1, 2$$

The equations (1.1)–(1.2) lead to the following Galerkin variational formulation:

Find $u\in V_1$, $v\in V_2$ such that

$$a(u, \tilde{u})+e_{12}(v, \tilde{u})=(f_1^*, \tilde{u})$$

(1.5) $$\forall\tilde{u}, \tilde{v}\in V_1\times V_2$$

$$e_{21}(u, \tilde{v})+b(v, \tilde{v})=(f_2^*, \tilde{v})$$

where $(\varphi, \psi)=\int_\Omega \varphi\psi\, d\Omega$.

The bilinear forms from (1.5) have the presentation

$$a(\varphi, \psi)=\int_\Omega\left[\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{1-\tilde{\nu}}{2}\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}\right]d\Omega,$$

(1.6) $$b(\varphi, \psi)=\int_\Omega\left[\frac{1-\tilde{\nu}}{2}\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}\right]d\Omega,$$

$$e_{12}(\varphi, \psi)=e_{21}(\psi, \varphi)=\int_\Omega\frac{1+\tilde{\nu}}{2}\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial y}d\Omega,$$

where $\varphi, \psi\in V_1\times V_2$.

Note that, if $\nu=0.5$ (or $\tilde{\nu}=1$; the material is incompressible), then the problem (1.5)–(1.6) is incorrect.

The remainder of this paper is organized as follows. In Section 2 the finite element problem is formulated. In Section 3 the proposed algebraic domain decomposition preconditioner is studied. Numerical tests demonstrating the behavior of the preconditioning relative condition number and the convergence of the related iterative methods are presented in Section 4.

2. Finite element approximation

From now we will assume that ω is a square mesh covering Ω and τ is the corresponding triangulation defined on ω. Let us $V_i^h \subset V_i$, $i=1, 2$, are the finite element spaces of piecewise linear functions with Lagrangian basis $\{\varphi_i\}_{i=1}^N$ and $\{\psi_i\}_{i=1}^N$, related to the triangulation τ. Then, the finite element numerical solution of the problem (1.5)-(1.6) can be described as follows:

Find $u^h=\Sigma_{i=1}^N u_i\varphi_i$, $v^h=\Sigma_{i=1}^N v_i\psi_i$ such that

$$\begin{aligned} &a(u^h, \varphi_i)+e_{12}(v^h, \varphi_i)=(f_1^*, \varphi_i) \\ &e_{21}(u^h, \psi_i)+b(v^h, \psi_i)=(f_1^*, \psi_i) \end{aligned} \qquad \forall i=1,\dots,N \tag{2.1}$$

The equations (2.1) can be written in the form

$$K\mathbf{w}=\mathbf{b}, \tag{2.2}$$

where K is the stiffness matrix and $\mathbf{w}$ is the vector of the nodal unknowns $\{u_i\}_{i=1}^N$ and $\{v_i\}_{i=1}^N$. The structure of the stiffness matrix depends on the ordering of the entries of the vector $\mathbf{w}$. If $\mathbf{w}^t=(u_1, u_2,\dots,u_N, v_1, v_2,\dots,v_N)$, then

$$K=\begin{bmatrix} K_{uu} & K_{uv} \\ K_{vu} & K_{vv} \end{bmatrix}. \tag{2.3}$$

The entries of K_{uu}, K_{vv} and K_{uv} are defined by (1.6), i. e.

$$k_{ij}^{uu}=a(\varphi_i, \varphi_j), \quad k_{ij}^{vv}=b(\psi_i, \psi_j), \quad k_{ij}^{uv}=e_{12}(\varphi_i, \psi_j) \tag{2.4}$$

The global matrix K can be assembled by the stiffness element matrices k_T using the usual finite element technology. For the considered model problem the matrix k_T takes the form (remember that the triangulation τ consists only of isosceles rectangular triangles)

$$k_T=\begin{bmatrix} a & -1 & -b & c & 0 & -c \\ -1 & 1 & 0 & -c & 0 & c \\ -b & 0 & b & 0 & 0 & 0 \\ c & -c & 0 & a & -b & -1 \\ 0 & 0 & 0 & -b & b & 0 \\ -c & c & 0 & -1 & 0 & 1 \end{bmatrix}, \tag{2.6}$$

where $a=(3-\tilde{\nu})/2$, $b=(1-\tilde{\nu})/2$, $c=(1+\tilde{\nu})/2$. These relations are used for local analyses of the relative spectral condition numbers of the preconditioning matrices in chapters 3 and 4.

3. Domain decomposition preconditioning

We are interested in the efficient iterative solution of the linear algebraic system (2.2). There are many recent papers dealing with investigating such kind of methods. In particular, we pay attention to iterative solvers based on the preconditioned conjugate gradient (PCG) method ([2], [8], [9], [10], [11]). The goal of this study is to construct a preconditioning matrix C satisfying the requirements:

— the number of arithmetic operations $\mathcal{N}$ for solving systems with the matrix C is considerably smaller, than the same number corresponding to the matrix K;

— the relative condition number $\varkappa=\varkappa(C^{-1}K)$ is as small as possible.

The global preconditioning strategy follows from the estimate

$$\mathsf{N}=0(\mathcal{N}\sqrt{\varkappa}\ln \varepsilon^{-1}), \tag{3.1}$$

where N is the total number of arithmetic operations needed to obtain an iterative solution of the system (2.2) with a relative error less than $\varepsilon>0$.

For simplicity of the presentation, we will assume from now, that the domain $\Omega=\cup_{i=1}^{p}\Omega_i$, where Ω_i are unit squares.

The approach of preconditioning the matrix K we have used is based on the Korn's inequality. This inequality related to the problem (1.5)-(1.6) is proved in [2]. It can be summarized by (3.2) as follows.

Let us denote by $\Phi(.,.)$ and $a_0(.,.)$ the bilinear forms defined by the relations

$$\Phi(\varphi,\ \psi)=a(\varphi,\ \varphi)+e_{12}(\varphi,\ \psi)+e_{21}(\varphi,\ \psi)+b(\psi,\ \psi)$$

$$a_0(\varphi,\ \psi)=\int_\Omega\left[\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}\right]d\Omega.$$

Then

$$\begin{aligned}&\alpha[a_0(\varphi,\ \varphi)+a_0(\psi,\ \psi)]\leqq\Phi(\varphi,\ \psi)\leqq\beta[a_0(\varphi,\ \varphi)+a_0(\psi,\ \psi)]\\&\forall(\varphi,\ \psi)\in V_1\times V_2,\end{aligned} \tag{3.2}$$

where

$$\alpha=(1-\tilde{\nu})/2,\quad \beta=(3+\tilde{\nu})/2. \tag{3.3}$$

Sometimes, it is more convenience to use the Korn's inequality in its matrix form

$$\alpha\xi^TR\xi\leq\xi^TK\xi\leq\beta\xi^TR\xi,\quad \forall\xi\in E^{2N} \tag{3.4}$$

where

$$R=\begin{bmatrix}A_0 & 0\\ 0 & A_0\end{bmatrix}$$

and the matrix A_0 is a finite element stiffness matrix corresponding to the bilinear form $a_0(.,.)$.

The matrix R can be considered as a preconditioning matrix for the stiffness matrix K obtained by an displacement by displacement decomposition. Now we use the domain decomposition technique from [5] for preconditioning the matrix A_0. After a reordering of the unknowns the matrix A_0 takes the following block form

$$(3.5)\qquad A_0=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}=\begin{bmatrix} A_{11} & 0 \\ A_{21} & S \end{bmatrix}\begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix},$$

where the first pivot block A_{11} corresponds to the inner nodes of the macroelements Ω_i and $S=A_{22}-A_{21}A_{11}^{-1}A_{12}$ is the Shur complement.

The presentation (3.5) has a fundamental meaning for constructing of domain decomposition preconditioners. The domain decomposition method can be considered as a generalization of the Schwartz algorithm for iterative solving of boundary value problems. Many authors in recent times deal with interpretations based on this idea (see for example in [1], [5], [7], [12], [13], [14]).

We study in this paper domain decomposition preconditioners defined by the relation

$$(3.6)\qquad C_0=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}=\begin{bmatrix} A_{11} & 0 \\ A_{21} & \tilde{B} \end{bmatrix}\begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix},$$

where $\tilde{B}$ is an approximation of the Shur complement S, satisfying the properties:

$$\begin{aligned} &\mathbf{x}^t\tilde{B}\,\mathbf{y}=\mathbf{y}^t\tilde{B}\,\mathbf{x} && \text{for any } \mathbf{x},\ \mathbf{y} \\ (3.7)\qquad &\tilde{B}\,\mathbf{e}=\emptyset && \mathbf{e}^t=(1,\dots 1) \\ &\mathbf{x}^t\tilde{B}\,\mathbf{x}>0 && \text{for any } \mathbf{x}\neq\mathbf{e}. \end{aligned}$$

Under the above assumptions the matrices $\tilde{B}$ and S are spectrally equivalent which leads to the spectrally equivalence of the matrices C_0 and A_0.

We consider now a family of preconditioning matrices where the block matrix $\tilde{B}$ from (3.6) is determined by the following relations.

Let us define at first the matrices $G_{i,k}$ related to the finite element approximation of the bilinear functional

$$g_{i,k}(\varphi,\psi)=\int_{\gamma_k}\frac{d\varphi}{ds}\,\frac{d\psi}{ds}\,ds,$$

where γ_k is the k-th edge $(k=1,\dots,4)$ of the subdomain Ω_i and where the boundary conditions from (3.4) are taken into account as well. The matrices $G_{i,k}$ are symmetric and positive semidefined. Therefore they can be written in the form

$$G_{i,k}=W_{i,k}^{-1}\,D_{i,k}\,W_{i,k},$$

where $W_{i,k}$ are the matrices of the eigenvectors of $G_{i,k}$ and $D_{i,k}$ are diagonal matrices containing the corresponding eigenvalues. Now we determine the matrices $\tilde{B}_{i,\alpha}$, $\alpha\geq 0$, as follows

$$\tilde{B}_{i,\alpha}=\bigoplus_{\gamma_k} G_{i,k,\alpha},$$

where $\oplus$ stands for assembling of matrices (this procedure is well known from the finite element method) and

$$G_{i,k,\alpha}=W_{i,k}^{-1}\ D_{i,k}^{\alpha}\ W_{i,k}.$$

Finally the preconditioner $C_{0,\alpha}$ is defined by (3.6), where $\tilde{B}$ is substituted by $\tilde{B}_{\alpha}$ and

$$\tilde{B}_{\alpha}=\bigoplus_{i} \tilde{B}_{i,\alpha}. \tag{3.8}$$

It follows from the above definition that the properties (3.7) are valid for the matrix $\tilde{B}_{\alpha}$. A more precise characterization of the proposed domain decomposition preconditioner is given by the next theorem.

Theorem 1. *The matrices A_0 and $C_{0,\alpha}$ are spectrally equivalent with positive scalars μ_1 and μ_2, i. e.*

$$\mu_1 \mathbf{v}^t\ C_{\alpha}\mathbf{v}\leqq \mathbf{v}^t\ A_0\mathbf{v}\leqq \mu_2\mathbf{v}^t\ C_{\alpha}\mathbf{v},\quad \text{for any } \mathbf{v}. \tag{3.9}$$

The relative spectral condition number $\varkappa(\alpha)=\mu_2/\mu_1$ is independent of the number of the subdomains.

P r o o f. The spectral equivalence, i. e. the existence of the positive constants μ_1 and μ_2 follows directly from (3.7). In order to prove the independence of $\varkappa(\alpha)$ of the number of the subdomains we will consider a constructive estimate for the relative condition number based on a local analysis on the level of macroelements.

It follows from the substructuring (3.5) of the stiffness matrix A_0, that the Shur complement S, can be assembled from the subdomain ones S_i, i. e.

$$S=\bigoplus_{i} S_i.$$

Now let us consider the generalized eigenvalue problem

$$S_i\mathbf{v}=\lambda^{(i)}\ \tilde{B}_{i,\alpha}\mathbf{v},\quad i=1,\dots,p. \tag{3.10}$$

We denote by $\lambda_{\min}^{(i)}$ and $\lambda_{\max}^{(i)}$ respectively the smallest and the largest eigenvalues of the problem (14) (the eigenvalue λ^* corresponding to the unit eigenvector is missed in this ordering). Then we have the estimation

$$\lambda_{\min}^{(i)}\mathbf{v}^t\ \tilde{B}_{i,\alpha}\mathbf{v}\leqq \mathbf{v}^t\ S_i\mathbf{v}\leqq \lambda_{\max}^{(i)}\mathbf{v}^t\ \tilde{B}_{i,\alpha}\mathbf{v}\quad \text{for any } \mathbf{v}. \tag{3.11}$$

Finally, taking into account (3.8), (3.10) and (3.11) we obtain (3.9) with constants $\mu_1=\lambda_{\min}$ and $\mu_2=\lambda_{\max}$, where $\lambda_{\min}$ and $\lambda_{\max}$ correspond to an arbitrary inner subdomain. ■

Now we define the preconditioning matrix of the stiffness matrix K from (2.2) as follows

$$C_\alpha = \begin{bmatrix} C_{o,\alpha} & 0 \\ 0 & C_{o,\alpha} \end{bmatrix} \tag{3.12}$$

Theorem 2. *The preconditioning matrix C_α and the stiffness matrix K are spectrally equivalent where the spectral condition number satisfies the inequality*

$$\varkappa(\alpha) = \varkappa(C_\alpha^{-1} K) \leq \mu_2/\mu_1 (3+\tilde{\nu})/(1-\tilde{\nu}) \tag{3.13}$$

Proof. The inequalities (3.13) follow directly from (3.2)-(3.4) and from Theorem 1. ■

Remark 1. A simple analysis shows that the total computational cost for performing one iterative step of the PCG method with preconditioner C_α is

$\mathcal{N} = 0(ph^{-2} \ln(h^{-1}))$, if a fast direct solver is used for solving the problems in the subdomain Ω_i,

$\mathcal{N} = 0(ph^{-2})$, if an optimal solver (e. g. multi grid or multi level) is used for solving the problems in the subdomains Ω_i.

Remark 2. An attractive feature of the presented domain decomposition algorithm is it's robust parallelisable, especially for distributed parallel computing systems. The communication cost and efficiency for such an approach applied to parallel iterative solving of second order boundary value problems are studied in [6].

4. Numerical tests

Firstly we consider the behavior of the condition number $\varkappa(\alpha)$, as a function of $h = 1/n$ (parameter of macroelement mesh refinement), of the modified Poisson's ratio $\tilde{\nu}$ and of the parameter α. In this purpose we use a local analysis on the level of the macroelements Ω_i. This analysis is based on the proof of Theorem 1. The generalized eigenvalue problems (3.10) are solved numerically by the subroutine F02BJF from the NAG library.

It is shown in Table 1 the growth of $\varkappa(\alpha)$ with $h = 1/n$. We have taken $\tilde{\nu} = 0.2$, the modified Poisson's ratio related to the concrete. The results are in agreement with the estimate $\varkappa(1) = O(h^{-1})$ ([1], [4], [5]). The behavior of $\varkappa(1/2)$ shows a logarithmic tendency of increase. The value of the parameter α obtained by a procedure of numerical minimization of $\varkappa(\alpha)$ is denoted by α^*. The values of α^* and $\varkappa(\alpha^*)$ are given in the last two columns. The numerical tests show that α^* tends to 0.5 with increase of n.

Table 1

Numerically computed relative condition numbers; $\varkappa = \varkappa(\alpha; n)$

n	$\varkappa(1/2)$	$\varkappa(1)$	α^*	$\varkappa(\alpha^*)$
2	11.83	10.56	0.906	10.47
3	15.22	16.12	0.697	14.55
4	17.81	21.60	0.617	17.43
5	19.94	27.06	0.574	19.72
6	21.77	32.51	0.547	21.65
7	23.40	37.96	0.531	23.34
8	24.87	43.39	0.520	24.84
9	26.21	48.84	0.511	26.19
10	27.44	54.27	0.505	27.43
11	28.58	59.71	0.501	28.57
12	29.64	65.14	0.500	29.64
13	30.65	70.58	0.500	30.65
14	31.59	76.01	0.500	31.59
15	32.46	81.43	0.500	32.46
16	33.26	86.84	0.500	33.26

In Table 2 the behavior of the relative spectral condition number for a fixed mesh parameter ($n=8$) as a function of the Poisson's ratio is given. The presented results show that the considered domain decomposition preconditioning algorithm is efficient for relatively small values of $\tilde{\nu} \in [0, 1)$. The case of almost incompressible materials ($\tilde{\nu}$ is near to 1) has to be treated by specialized methods and algorithms (see for example in [10], [11]).

Table 2

Numerically computed relative condition numbers; $\varkappa = \varkappa(\alpha; \tilde{\nu})$

$\tilde{\nu}$	$\varkappa(1/2)$	$\varkappa(1)$
−1.	9.07	11.64
0.1	21.82	37.49
0.2	24.87	43.39
0.3	28.80	50.99
0.4	34.04	61.13
0.5	41.39	75.33
0.6	52.43	96.62
0.7	70.82	132.12
0.8	107.59	203.11
0.9	217.79	416.08

Now we will consider the numerical solution of the problem (2.2), where the PCG method with a preconditioner C_α is used. The stopping criterion is 10^{-8}. The domain is shown in Fig. 1.

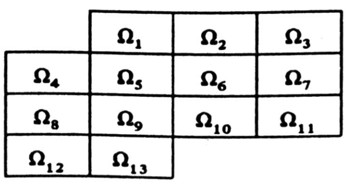

Fig. 1
Domain geometry of the test problem

Table 3

Number of iterations of the considered PCG algorithm

n	$N(0.5)$	$N(1)$	$N(\alpha^*)$
4	9	10	8
8	10	12	9
16	11	17	11

The number of the iterations $N(\alpha; n)$ needed to reach an iteration relative error less than 10^{-8} are presented in Table 3. Here as well as in Table 1 we have taken $\tilde{\nu}=0.2$. The results show a nearly optimal i. e. $O(\ln(h^{-1}))$ number of iterations for $\alpha=0.5$.

Let us note again that the number of iteration is independent of the number of the subdomains p.

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