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Some Remarks on Inequalities of L. Iliev

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Presented by Bl. Sendov

L. Iliev proved that if sequence $(a_n)_{n=0}^{\infty} \in \alpha$ then following inequalities hold:

$$a_n^2 - a_{n-1} a_{n+1} \geqq 0 \qquad n = 1, 2, \dots$$

$$a_n^2 - a_{n-2} a_{n+2} \geqq 0 \qquad n = 2, 3, \dots$$

$$\left(\frac{a_{n-3}}{a_n}\right)^2 - \left(\frac{a_{n-2}}{a_n}\right)^3 \leqq 0 \quad n = 3, 4, 5 \dots, \quad a_n \neq 0.$$

In this paper we give a generalization of these inequalities and we will show that first inequality implies other.

1. Introduction

The sequence $(a_n)_{n=0}^{\infty}$ belongs to class α if for any polynomial $b(z) = b_0 + b_1 z + \dots + b_n z^n$ which has only real zeros, the polynomial $b(z) * (a_n) = b_0 a_0 + b_1 a_1 z + \dots + b_n a_n z^n$ also has only real zeros.

We know some simple properties of sequence $(a_n)_n$.

(i) If $(a_n)_n \in \alpha$, $a_\lambda \neq 0$ and $a_{\lambda+\mu} \neq 0$ then $a_{\lambda+k} \neq 0$ for all $k = 0, 1, 2, \dots, \mu$ and either they have same signs or form alternative sequence.

(ii) For $(a_n)_n \in \alpha$ following inequalities are valid:

(1) $$a_n^2 - a_{n-1} a_{n+1} \geqq 0 \qquad n = 1, 2, \dots$$

(2) $$a_n^2 - a_{n-2} a_{n+2} \geqq 0 \qquad n = 2, 3, \dots$$

(3) $$\left(\frac{a_{n-3}}{a_n}\right)^2 - \left(\frac{a_{n-2}}{a_n}\right)^3 \leqq 0 \quad n = 3, 4, 5 \dots, a_n \neq 0$$

(iii) If $(a_n)_{n=0}^{\infty} \in \alpha$ and $(a'_n)_{n=0}^{\infty} \in \alpha$ then $(a_n a'_n)_{n=0}^{\infty} \in \alpha$ and $(a_n)_{n=\nu}^{\infty} \in \alpha$.

2.

Here, we shall prove the following result.

Theorem 1. *Let* $(a_n)_{n=0}^{\infty} \in \alpha$, $a_n \neq 0$ *for all* $n \in \mathbb{N}$. *Let* x_i, y_i *be integers*, p_i *real*, $i=1, 2 \ldots n$ *and* $x_1 \geqq x_2 \geqq \ldots \geqq x_n$, $y_1 \geqq y_2 \geqq \cdots \geqq y_n$.

$$\prod_{r=1}^{n} |a_{x_r}|^{p_r} \geqq \prod_{r=1}^{n} |a_{y_r}|^{p_r} \tag{4}$$

is valid if and only if

$$\sum_{r=1}^{k} p_r x_r \leqq \sum_{r=1}^{k} p_r y_r \quad k=1, 2, \ldots, n-1 \tag{5}$$

and

$$\sum_{r=1}^{n} p_r x_r = \sum_{r=1}^{n} p_r y_r. \tag{6}$$

Proof. Let $\gamma_n = -\ln |a_n|$. From (1) is obviously that $(\gamma_n)_n$ is convex sequence. By the Majorization theorem, [2], conditions (5) and (6) are equivalent to inequality

$$\sum_{r=1}^{n} p_r f(x_r) \leqq \sum_{r=1}^{n} p_r f(y_r)$$

where f is any convex function. But, if $(\gamma_n)_n$ is convex sequence then function f whose graph is the polygonal line with corner points (n, γ_n), $n \in \mathbb{N}$, is convex on $[1, \infty)$, [4], and we have

$$\sum_{r=1}^{n} p_r \gamma_{x_r} \leqq \sum_{r=1}^{n} p_r \gamma_{y_r}.$$

This inequality is equivalent to (4). ■

Specially, if we put $p_1=q-p$, $p_2=r-q$, $x_1=x_2=q$, $y_1=r$, $y_2=p$ where p, q, r are integers and $1 \leqq p<q<r$ then we have

$$|a_q|^{r-p} \geqq |a_p|^{r-q} \cdot |a_r|^{q-p}. \tag{7}$$

If $q=n$, $p=n-i$, $r=n+i$ then (7) becomes

$$|a_n|^{2i} \geqq |a_{n-i}|^i \, |a_{n+i}|^i$$

$$\text{i. e. } a_n^2 \geqq a_{n-i} a_{n+1}$$

what is the generalization of (1) and (2).

If $r-q=2k$ then (7) becomes

$$\left|\frac{a_p}{a_r}\right|^{2k} \leqq \left|\frac{a_{r-2k}}{a_r}\right|^{r-p}$$

and for $k=1$, $r=n$ and $p=n-3$ we get inequality (3).

If $r-p=2k$ then from (7) we have

$$\left|\frac{a_{r-2k}}{a_r}\right|^{r-q} \leqq \left|\frac{a_q}{a_r}\right|^{2k}$$

and for $k=2$, $r=n$ and $q=n-3$ we have

$$\left(\frac{a_{n-4}}{a_n}\right)^3-\left(\frac{a_{n-3}}{a_n}\right)^4\leqq 0$$

what is some kind of generalization of inequality (3).

If we put $p_1=p_2=1$, $x_1=n+k$, $x_2=n-k$, $y_1=n+k+1$, $y_2=n-(k+1)$, $k=0$, 1, 2,..., $n-1$ then from Theorem 1 we can get following inequality sequence:

$$a_n^2\geqq a_{n-1}a_{n+1}\geqq a_{n-2}a_{n+2}\geqq\dots\geqq a_{n-k}a_{n+k}\geqq\dots\geqq a_{2n}a_0.$$

3.

Of course, we can use some other results for convex sequences. For example, [3]:

If $(\gamma_n)_n$ is convex sequence then inequality $\Sigma_{i=1}^n p_i\gamma_i\geqq 0$ holds if and only if p_i are real numbers which satisfy following conditions:

(8) $$\sum_{i=1}^{n} p_i=0,\quad \sum_{i=1}^{n} ip_i=0$$

(9) $$\sum_{i=k}^{n}(i-k+1)p_i\geqq 0\quad k=3, 4,\dots,n.$$

In the term of sequence $(a_n)_n$ we have the following statement.

Theorem 2. *Let $p_1, p_2,\dots p_n$ be real numbers. The inequality $\Pi_{i=1}^n |a_i|^{p_i}\leqq 1$ holds for every $(a_n)_n\in\alpha$ if and only if p_i satisfy* (8) *and* (9).

Specially, if we put $p_{2k+1}=\frac{1}{n+1}(k=0, 1,\dots n)$, $p_{2k}=-\frac{1}{n}, (k=1, 2,\dots,n)$ then we have

$$\sqrt[n]{|a_2a_4\dots a_{2n}|}\geqq\sqrt[n+1]{|a_1a_3\dots a_{2n+1}|}.$$

(Note that is Nanson's inequality, [4], applied on sequence $\gamma_n=-\ln|a_n|$.)

4. Remarks

Condition ($a_n\neq 0$, $\forall n\in\mathrm{N}_0$) can be replaced by ($a_n\neq 0$, for $n\in I$) where $I=\{k, k+1, k+2,\dots,k+l\}$, $k, l\in\mathrm{N}_0$.

5. Application

Let $f_k(z)=b_k z^{m_k}e^{\lambda_k z}\Pi_{n\in J_k}(1+\frac{z}{z_n})$ be entire function where $z_n>0$ for $n\in J_k$,

$b_k \in \mathrm{R}$, $m_k \in \mathrm{N}_0$, $\lambda_k \geqq 0$ and $\Sigma_{n\in J_k} \frac{1}{z_n} < \infty$, and let $x_k \geqq 0$, $k=1, 2, \ldots, s$.

Then we can define following functions:

$$Q^{(s)}(z) = \prod_{k=1}^{s} f_k(x_k z) = \sum_{n=0}^{\infty} Q_n(x_1, \ldots x_s) \frac{z^n}{n!}$$

$$R^{(s)}(z) = \prod_{k=1}^{p} f_k(x_k z) \prod_{k=p+1}^{s} f_k(x_k + z) = \sum_{n=0}^{\infty} R_n(x_1, \ldots x_s) \frac{z^n}{n!}$$

$$S^{(s)}(z) = \prod_{k=1}^{s} f_k(x_k + z) = \sum_{n=0}^{\infty} S_n(x_1, \ldots x_s) \frac{z^n}{n!}.$$

L. Iliev, [1], proved that sequences $(Q_n(x_1, \ldots x_s))_n$, $(R_n(x_1, \ldots x_s))_n$ and $(S_n(x_1, \ldots x_s))_n$ belong to class α and for these sequences hold all previous inequalities. For example:

$$Q_n^2(x_1, \ldots x_s) \geqq Q_{n-i}(x_1, \ldots x_s)\, Q_{n+i}(x_1, \ldots x_s)$$

$$|Q_q^{r-p}(x_1, \ldots x_s)| \geqq |Q_p^{r-q}(x_1, \ldots x_s)|\; |Q_r^{q-p}(x_1, \ldots x_s)|$$

$$Q_{n-k}(x_1, \ldots x_s)\, Q_{n+k}(x_1, \ldots x_s) \geqq Q_{n-i}(x_1, \ldots x_s)\, Q_{n+i}(x_1, \ldots x_s) \quad k < i.$$

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