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On a Type of Sasakian Manifold

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In this paper concircularly flat Sasakian manifold and a Sasakian manifold satisfying $R(X, Y)..C^*=0$ have been studied.

Introduction

Let M^n be a contact Riemannian manifold with a contact form η, the associated vector field ξ, $(1-1)$ tensor field Φ and the associated Riemannian metric g. If ξ is a Killing vector field, then M^n is called a K-contact Riemannian manifold [1], [2]. A K-contact Riemannian manifold is called Sasakian [2] if

$$(\nabla_X \Phi)(Y)=g(X, Y)\xi-\eta(Y)X, \tag{1}$$

holds, where ∇ denotes the operator of covariant differentiation with respect to g. This paper deals with a type of Sasakian manifold in which

$$R(X, Y).C^*=0 \tag{2}$$

where C^* is the concircular curvature tensor [3] defined by

$$C^*(X, Y)Z=R(X, Y)Z-\frac{r}{n(n-1)}[g(Y, Z)X-g(X,Z)Y], \tag{3}$$

R is the curvature tensor, r is the scalar curvature and $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. In this connection we mention the works of K. Sekigawa [4], Z. I. Szabo [5], L. Verstraelen [6], M. Petrovic-Torgasev and L. Verstraelen [7] who studied Riemannian manifolds or hypersurfaces of such manifolds satisfying the conditions similar to it. It is easy to see that $R(X, Y).R=0$ implies $R(X, Y).C^*=0$. So it is meaningful to undertake the study of manifolds satisfying the condition (2).

In this paper it is proved that if a Sasakian manifold $M^n (n\geqq 2)$ is concircularly flat then it is locally isometric with a unit sphere $S^n(1)$. Also it is proved that if in a Sasakian manifold the relation (2) holds, then it is also locally isometric with a unit sphere $S^n(1)$.

1. Preliminaries

Let S denotes the Ricci tensor of type (0,2) of M^n. It is known that in a Sasakian manifold M^n, besides the relation (1) the following relations hold [1], [2], [8]

$$\Phi(\xi)=0 \tag{1.1}$$

$$\eta(\xi)=1 \tag{1.2}$$

$$\Phi^2(X)=-X+\eta(X)\xi \tag{1.3}$$

$$g(\Phi X, \Phi Y)=g(X, Y)-\eta(X)\eta(Y) \tag{1.4}$$

$$g(\xi, X)=\eta(X) \tag{1.5}$$

$$\nabla_X \xi=-\Phi X \tag{1.6}$$

$$S(X, \xi)=(n-1)\eta(X) \tag{1.7}$$

$$g(R(\xi, X)Y, \xi)=g(X, Y)-\eta(X)\eta(Y) \tag{1.8}$$

$$R(\xi, X)\xi=-X+\eta(X)\xi \tag{1.9}$$

and

$$(\nabla_X \Phi)(Y)=R(\xi, X)Y \tag{1.10}$$

for any vector fields X, Y.

The above results will be used in the next section.

2. Sasakian manifold satisfying $C^*(X, Y)Z=0$

Let us suppose that in a Sasakian manifold

$$C^*(X, Y)Z=0. \tag{2.1}$$

Then

$$R(X, Y)Z=\frac{r}{n(n-1)}[g(Y, Z)X-g(X,Z)Y] \tag{2.2}$$

or,

$$g(R(X, Y)Z, W)=\frac{r}{n(n-1)}[g(Y, Z)g(X, W)-g(X, Z)g(Y, W)]. \tag{2.3}$$

Putting $X=W=\xi$ in (2.3) we get

$$g(R(\xi, Y)Z, \xi)=\frac{r}{n(n-1)}[g(Y, Z)g(\xi, \xi)-g(\xi, Z)g(Y, \xi)]$$

or,

$$g(Y, Z)-\eta(Y)\eta(Z)=\frac{r}{n(n-1)}[g(Y, Z)-\eta(Y)\eta(Z)],$$

by (1.8), (1.2), (1.5)
or,

$$[\frac{r}{n(n-1)}-1][g(Y, Z)-\eta(Y)\eta(Z)]=0.$$

Then either

$$r=n(n-1) \text{ or, } g(Y, Z)=\eta(Y)\eta(Z).$$

Now if $g(Y, Z)=\eta(Y)\eta(Z)$, then from (1.4) we get

$$g(\Phi Y, \Phi Z)=0$$

which is not possible

Therefore $r=n(n-1)$.

Now putting the value of r in (2.2) we get the manifold is of constant curvature unity. Hence we can state the following theorem:

Theorem 1. *A concircularly flat Sasakian manifold $M^n(n\geqq 2)$ is locally isometric with a unit sphere $S^n(1)$.*

3. Sasakian manifold satisfying $R(X, Y).C^*=0$

We have

$$\eta(C^*(X, Y)Z)=g(C^*(X, Y)Z, \xi)$$

$$=g(R(X, Y)Z-\frac{r}{n(n-1)}[g(Y, Z)X-g(X,Z)Y], \xi)$$

$$=-[\eta(Y)g(Z, X)-\eta(X)g(Z, Y)]$$

$$-\frac{r}{n(n-1)}[\eta(X)g(Y, Z)-\eta(Y)g(X, Z)],$$

by (1.5), (1) and (1.10)
or,

$$(3.1)\qquad \eta(C^*(X, Y)Z)=[\frac{r}{n(n-1)}-1][\eta(Y)g(Z, X)-\eta(X)g(Z, Y)].$$

Putting $X=\xi$ in (3.1) we get

$$(3.2)\qquad \eta(C^*(\xi, Y)Z)=[\frac{r}{n(n-1)}-1][\eta(Z)\eta(Y)-g(Y, Z)].$$

Again putting $Z=\xi$ in (3.1) we get

$$\eta(C^*(X, Y)\xi)=0. \tag{3.3}$$

Now

$$(R(X, Y).C^*)(U, V)W=R(X, Y)C^*(U, V)W-C^*(R(X, Y)U, V)W$$
$$-C^*(U, R(X, Y)V)W-C^*(U, V)R(X, Y)W.$$

In virtue of (2) we get

$$R(X, Y)C^*(U, V)W-C^*(R(X, Y)U, V)W-C^*(U, R(X, Y)V)W$$
$$-C^*(U, V)R(X, Y)W=0. \tag{3.4}$$

Therefore

$$g[R(\xi, Y)C^*(U, V)W, \xi]-g[C^*(R(\xi, Y)U, V)W, \xi]$$
$$-g[C^*(U, R(\xi, Y)V)W, \xi]-g[C^*(U, V)R(\xi, Y)W, \xi]=0.$$

From this it follows that

$$C^*(U, V, W, Y)-\eta(Y)\eta(C^*(U, V)W)+\eta(U)\eta(C^*(Y, V)W)$$
$$+\eta(V)\eta(C^*(U, Y)W)+\eta(W)\eta(C^*(U, V)Y)-g(Y, U)\eta(C^*(\xi, V)W)$$
$$-g(Y, V)\eta(C^*(U, \xi)W)-g(Y, W)\eta(C^*(U, V)\xi)=0, \tag{3.5}$$

where $g(C^*(U, V)W, Y)='C^*(U, V, W, Y)$.

Putting $Y=U$ in (3.5) we get

$$C^*(U, V, W, U)-\eta(U)\eta(C^*(U, V)W)+\eta(U)\eta(C^*(U, V)W)$$
$$+\eta(V)\eta(C^*(U, U)W)+\eta(W)\eta(C^*(U, V)U)-g(U, U)\eta(C^*(\xi, V)W)$$
$$-g(U, V)\eta(C^*(U, \xi)W)-g(U, W)\eta(C^*(U, V)\xi)=0. \tag{3.6}$$

Let $\{e_i\}$, $i=1, 2,\dots, n$ be an orthonormal basis of the tangent space at any point. Then the sum for $1\leqq i\leqq n$ of the relation (3.6) for $U=e_i$ gives

$$\eta(C^*(\xi, V)W)=\frac{1}{n-1}[S(V, W)-\frac{r}{n}g(V, W)$$
$$+(\frac{r}{n(n-1)}-1)(n-1)\eta(W)\eta(V)]. \tag{3.7}$$

Using (3.1) and (3.7) it follows from (3.5) that

$$'C^*(U, V, W, Y)+\frac{r}{n(n-1)}g(Y, U)g(V, W)-\frac{r}{n(n-1)}g(U, W)g(Y, V)$$
$$-\frac{1}{(1-n)}[S(U, W)g(Y, V)-S(V, W)g(Y, U)]=0. \tag{3.8}$$

From (3.2) and (3.7) we get

$$S(Y, Z)=(n-1)\,[g(Y, Z)]. \tag{3.9}$$

Using (3.9), the relation (3.8) reduces to

$$'C^*(U, V, W, Y)=[\frac{r}{n(n-1)}-1]\,[g(Y, V)g(U, W)-g(Y, U)g(V, W)]. \tag{3.10}$$

From (3) and (3.10) we get

$$'R(U, V, W, Y)=[g(Y, U)g(V, W)-g(Y, V)g(U, W)], \tag{3.11}$$

where

$$'R(U, V, W, Y)=g(R(U, V)W, Y).$$

Hence we can state the following theorem:

Theorem 2. *If in a Sasakian manifold $M^n(n\geqq 2)$ the relation $R(X, Y).C^*=0$ holds, then it is locally isometric with a unit sphere S^n* (1).

For a concircularly symmetric Riemannian manifold we have $\nabla C^*=0$. Hence for such a manifold $R(X, Y).C^*=0$ holds. Thus we have the following Corollary of the above theorem:

Corollary. *A concircularly symmetric Sasakian manifold $M^n(n\geqq 2)$ is locally isometric with a unit sphere S^n* (1).

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