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On Hadamard's Inequalities for Convex Functions

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Some refinements of the well-known Hadamard's inequalities for convex functions are given.

1. Introduction

Let $f: I \subseteq \mathsf{R} \to \mathsf{R}$ be a convex function on interval I and $a, b \in I$ with $a<b$. The double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is known in literature as Hadamard's inequalities. We note that, J. Hadamard was not the first who discovered them. As is pointed out by D. S. Mitrinović and I. B. Lacković [9] the inequalities (1) are due to C. Hermite who discovered them in 1883, ten years before J. Hadamard [5].

In this paper we will give some improvements of this classic fact.

2. The main results

The following refinement of the first inequality in (1) holds.

Theorem 1. *Let f be as above and $\alpha, \beta: [a, b] \to \mathsf{R}_+$ be two continuous mappings so that $\alpha(x)+\beta(y)>0$ for all x, y in $[a, b]$. Then one has the inequalities*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)\mathrm{d}x\,\mathrm{d}y \tag{2}$$

$$\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b \frac{1}{2}\left[f\left(\frac{\alpha(x)x+\beta(y)y}{\alpha(x)+\beta(y)}\right)+f\left(\frac{\beta(y)x+\alpha(x)y}{\alpha(x)+\beta(y)}\right)\right]dxdy$$

$$\leq \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Proof. The Jensen's inequality for double integrals yields that

$$f\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b\left(\frac{x+y}{2}\right)dxdy\right)\leqq\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy$$

which shows the first inequality in (2).

By the convexity of f on $[a, b]$ one has:

$$f\left(\frac{x+y}{2}\right)\leqq\frac{1}{2}\left[f\left(\frac{\alpha(x)x+\beta(y)y}{\alpha(x)+\beta(y)}\right)+f\left(\frac{\beta(y)x+\alpha(x)y}{\alpha(x)+\beta(y)}\right)\right]$$
$$\leqq\frac{f(x)+f(y)}{2}$$

for all x, y in $[a, b]$. Integrating these inequalities on $[a, b]^2$ we obtain the second part of (2).

Now, for a given convex mapping $f:[a, b]\to\mathbf{R}$, let $H:[0, 1]\to\mathbf{R}$ be defined by:

$$H(t):=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)\mathrm{d}x.$$

The following theorem holds [3]

Theorem 2. *In the above assumptions, we have:*
(*i*) *H is convex on* $[0, 1]$;
(*ii*) $\inf_{t\in[0,1]} H(t)=H(0)=f\left(\frac{a+b}{2}\right)$;
(*iii*) $\sup_{t\in[0,1]} H(t)=H(1)=\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$;
(*iv*) *H is monotonous nondecreasing on* $[0, 1]$.

Proof. "(i)". It is obvious.
"(ii), (iii)". We will prove the following inequalities:

$$(3)\qquad f\left(\frac{a+b}{2}\right)\leqq H(t)\leqq t.\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x+(1-t).f\left(\frac{a+b}{2}\right)$$
$$\leqq\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x,\ t\in[0, 1].$$

By Jensen's integral inequality, we have:

$$H(t)\geqq f\left(\frac{1}{b-a}\int_a^b\left[tx+(1-t)\frac{a+b}{2}\right]\mathrm{d}x\right)=f\left(\frac{a+b}{2}\right).$$

The other inequalities in (3) are obvious from the convexity of f.

"(iv)". Let $t_1, t_2 \in (0, 1)$ with $t_2 > t_1$. Then, from H being convex on $(0, 1)$, one has:

$$(H(t_2)-H(t_1))/(t_2-t_1) \geqq H'_+(t_1)$$

$$=\frac{1}{b-a}\int_a^b f'_+\left[t_1 x+(1-t_1)\frac{a+b}{2}\right]\left(x-\frac{a+b}{2}\right)dx$$

$$\geqq \frac{1}{t_1}\left[H(t_1)-f\left(\frac{a+b}{2}\right)\right] \geqq 0.$$

Consequently, $H(t_2)-H(t_1) \geqq 0$ for $1 \geqq t_2 > t_1 \geqq 0$ and the statement is proven.

Now, we will define the second mapping in connection to Hadamard's inequalities. Let $f:[a, b] \to \mathrm{R}$ be convex on $[a, b]$. Put

$$F:[0, 1] \to \mathrm{R},\ F(t):=\frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,dx\,dy.$$

The following theorem holds [3]:

Theorem 3. *Let* $f:[a, b] \to \mathrm{R}$ *be as above. Then*:

(i) $F\left(s+\frac{1}{2}\right)=F\left(\frac{1}{2}-s\right)$ *for all* s *in* $[0, \frac{1}{2}]$;

(ii) F *is convex on* $[0, 1]$;

(iii) We have:

$$\sup_{t\in[0, 1]} F(t)=F(0)=F(1)=\frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$\inf_{t\in[0, 1]} F(t)=F\left(\frac{1}{2}\right)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dx\,dy;$$

(iv) F *is monotonous nonincreasing on* $[0, 1/2]$ *and nondecreasing on* $[1/2, 1]$;

(v) One has the inequality:

$$H(t) \leqq F(t) \text{ for all } t \text{ in } [0, 1]. \tag{4}$$

Proof. "(i), (ii)". It is obvious.

"(iii)" Follows by Theorem 1 for $\alpha(x)=t$, $\beta(y)=1-t$, x, y in $[a, b]$ and $t \in [0, 1]$.

"(iv)". Since F is convex on $(0, 1)$, we have for $t_2 > t_1$, $t_2, t_1 \in (1/2, 1)$:

$$(F(t_2)-F(t_1))/(t_2-t_1) \geqq F'_+(t_1)$$

$$=\frac{1}{(b-a)^2}\int_a^b\int_a^b f'_+(t_1 x+(1-t_1)y)(x-y)\,dx\,dy$$

$$\geqq \frac{2}{2t_1-1}\left(F(t_1)-F\left(\frac{1}{2}\right)\right) \geqq 0,$$

which shows that F is monotonous nondecreasing on [1/2, 1].

The second part follows by "(i)".

"(v)". We have:

$$H(t) := \frac{1}{b-a}\int_a^b f\left(\frac{1}{b-a}\int_a^b [tx+(1-t)y]\,dy\right)dx.$$

Using Jensen's integral inequality, we derive easily (4).

In the recent paper [4] the following refinement of Hadamard's inequalities for multiple integrals is proved:

Theorem 4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping on the interval I and $a, b \in I$ with $a<b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)^k}\int_a^b \dots \int_a^b f\left(\frac{x_1+\dots+x_k}{k}\right)dx_1\dots dx_k$$

$$\leq \frac{1}{(b-a)^{k-1}}\int_a^b \dots \int_a^b f\left(\frac{x_1+\dots+x_{k-1}}{k-1}\right)dx_1\dots dx_{k-1} \leq \dots$$

$$\leq \frac{1}{(b-a)^2}\int_a^b \int_a^b f\left(\frac{x_1+x_2}{2}\right)dx_1\,dx_2$$

for all k a natural number with $k \geq 3$.

Now, we will give another result of this type.

Theorem 5. *In the above assumptions, we have:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)^n}\int_a^b \dots \int_a^b f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)dx_1\dots dx_n \tag{5}$$

$$\leq \frac{1}{b-a}\int_a^b f(x)\,dx$$

for all $p_i \geq 0$ $(i=1,\dots,n)$ with $P_n := \sum_{i=1}^n p_i > 0$ and n is a positive integer.

Proof. By Jensen's integral inequality for multiple integrals one has:

$$f\left(\frac{1}{(b-a)^n}\int_a^b \dots \int_a^b \left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)dx_1\dots dx_n\right)$$

$$\leq \frac{1}{(b-a)^n}\int_a^b \dots \int_a^b f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)dx_1\dots dx_n$$

for where results the first part of (5).

The second inequality in (5) follows from Jensen's discrete inequality:

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leqq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i),$$

by integration on $[a, b]^n$.

Now, we will point out some inequalities of Hadamard's type for differentiable convex functions.

Theorem 6. *Let $f : I \subseteq \mathrm{R} \to \mathrm{R}$ be a differentiable convex functions on I and $a, b \in I$ with $a<b$. Then one has the inequalities:*

$$(6) \qquad 0 \leqq \frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,dx\,dy$$

$$\leqq t\left(\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right)$$

for all t in $[0, 1]$.

Proof. The first inequality follows from Theorem 3.

To prove the second part of (6), we observe that:

$$f(tx+(1-t)y) - f(y) \geqq t(x-y)f'(y),$$

for all x, y in $[a, b]$ and t in $[0, 1]$. Integrating this inequality on $[a, b]^2$ a simple computation shows that the statement is true.

Another result which is interesting by its consequences, is the following:

Theorem 7. *Let f be as above. Then:*

$$f(t)+f'(t)\left(\frac{a+b}{2}-t\right) \leqq \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\leqq \frac{f(t)}{2} + \frac{1}{2}\cdot\frac{bf(b)-af(a)-t(f(b)-f(a))}{b-a}$$

for all t in $[a, b]$.

Proof. The first inequality is obvious.

The second inequality follows by

$$f(t)-f(x) \geqq (t-x)f'(x), \quad x,\ t \in [a, b]$$

by integration over x on $[a, b]$.

Corollaries. a. *In the above assumptions (for $0 \leqq a < b$), we have:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leqq \min\{H_f(a, b), G_f(a, b), A_f(a, b)\}$$

where

$$H_f(a, b) := \frac{1}{2}[f(\frac{2ab}{a+b}) + \frac{bf(b)+af(a)}{b+a}],$$

$$G_f(a, b) := \frac{1}{2}[f(\sqrt{ab}) + \frac{\sqrt{b}f(b)+\sqrt{a}f(a)}{\sqrt{b}+\sqrt{a}}],$$

$$A_f(a, b) := \frac{1}{2}[f(\frac{a+b}{2}) + (f(a)+f(b))/2]$$

(see also [10]).

b. *If* $f'(\sqrt{ab}) \geqq 0$, *then*:

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geqq f(\sqrt{ab}).$$

c. *If* $f'(\frac{2ab}{a+b}) \geqq 0$, *then*:

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geqq f(\frac{2ab}{a+b}).$$

d. *We have the inequality*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geqq \max\{f(a)+f'(a)\frac{b-a}{2}, f(b)+f'(b)\frac{a-b}{2}\}.$$

e. *We have the inequalities:*

$$0 \leqq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \leqq \frac{f'(b)-f'(a)}{2}(b-a)$$

f. *Suppose that* p_i *are nonnegative real numbers with* $p_i > 0$ *and* x_i *are in* $[a, b]$ $(i=1,\dots,n)$ *so that:*

$$\frac{a+b}{2}\sum_{i=1}^{n} f'(x_i)p_i \geqq \sum_{i=1}^{n} p_i f'(x_i)x_i.$$

Then one has the inequality

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geqq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i).$$

Remark 1. If we choose in the above inequalities $f(x) := x^p$, $x \in [a, b]$ $(0 \leqq a < b,\ p > 1)$ or $f(x) := 1/x$, $x \in [a, b]$ $(0 < a < b)$ we can obtain some inequalities improving the results established in [10] (see also the references listed in [10]).

In paper [4] it is also proved the following discrete analogue of Hadamard's inequalities:

Theorem 8. *Let f be a convex mapping on I and a, b belong to I. Then the following inequalities are true*

$$f\left(\frac{a+b}{2}\right) \leqq \frac{1}{n}\sum_{i=1}^{n} f\left[\left(\frac{i}{n+1}\right)a + \left(1-\frac{i}{n+1}\right)b\right] \leqq \frac{f(a)+f(b)}{2}$$

for all positive integer n.

Now, we will give another result of this type.

Theorem 9. *Let f be as above, $a, b \in I$, $p_i \geqq 0$ $(i=1,\ldots,n)$ with $P_n > 0$ and t is fixed in* [0, 1]. *If t_i are in* [0, 1] *so that $\frac{1}{P_n}\sum_{i=1}^{n} p_i t_i = t$, then one has the inequalities:*

$$f\left(\frac{a+b}{2}\right) \leqq \frac{1}{2}[f(ta+(1-t)b)+f((1-t)a+tb)] \tag{7}$$

$$\leqq \frac{1}{P_n}\sum_{i=1}^{n} p_i \frac{1}{2}[f(t_i a+(1-t_i)b)+f((1-t_i)a+t_i b)]$$

$$\leqq \frac{f(a)+f(b)}{2}.$$

Proof. The first inequality is obvious from the convexity of f. Let us consider the mapping:

$$g : [0, 1] \to \mathrm{R},\quad g(t) := \frac{1}{2}[f(ta+(1-t)b)+f((1-t)a+tb)].$$

It is clear that g is convex on [0, 1] and then Jensen's discrete inequality for g yields the second part of (7).

The last part is also obvious.

For other inequalities in connection to Hadamard's result, see [1-12] where further references are given.

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