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On Probable Error of the Monte Carlo Method for Numerical Integration

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We consider the problem of computation of the definite integral $I=\int_a^b f(x)q(x)\,dx$, *where* $q(x)$ is a density function ($q(x)\geqq 0$ and $\int_a^b q(x)\,dx=1$). The aim of this article is to estimate the probable error by integral modulus of continuity.

1. Introduction

Let denote by $N(\mu, \sigma^2)$ a normal random variable with mean μ and variance σ^2, which is understood to be the constant μ if $\sigma=0$. $\Phi(x)=\frac{1}{2\pi^{1/2}}\int_{\infty}^{x} e^{-t^2/2}dt$ is a distribution function of $N(0, 1)$; $\chi(M)$ is a characteristic function of the set M; dx – Lebesque measure; $C[a, b]$ is the set of real-valued continuous functions on the segment $[a, b]$, equipped with the uniform norm $\|\cdot\|$.

Let I be any functional that we estimate by Monte-Carlo method; θ_n be the estimator, where n is the number of trials.

Definition 1. We define the probable error as follows

$$P\{|I-\theta_n|\geqq r_n\}=1/2=P\{|I-\theta_n|\leqq r_n\}. \tag{1}$$

If the standard deviation $\sigma(\theta_n)<\infty$, the normal convergence in the Central Limit Theorem holds.

$$P\{|I-\theta_n|\leqq x\sigma(\theta_n)n^{-1/2}\}\approx\Phi(x).$$

Obviously, from (1) and $\Phi(0.6745)=\frac{1}{2}$ we have

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(2) $$r_n = 0.6754\,\sigma(\theta_n) n^{-1/2}.$$

In other words we obtain the upper estimate for r_n with a confidence coefficient $\frac{1}{2}$. If in the Central Limit Theorem the convergence in distribution is not to the normal distribution, the estimate (2) has no sense, since $\sigma(\theta_n)=\infty$. In the case when the Monte-Carlo method has a probable error $r_n = 0(n^{-1/2-\varepsilon})$ where $\varepsilon>0$, then we say that the Monte-Carlo method has an overconvergent probable error.

Let the segment $[a, b]$ be separated to closed subintervals Δ_j, $j=1,2,\dots,n$ such that:

(3) $[a, b] = \bigcup_{j=1}^{n} \Delta_j$ and for the open intervals $\Delta_i^0 \cap \Delta_j^0 = \emptyset$,

for $i \neq j$ $(i, j=1,\dots,n)$;

(4) for the uniform distribution of probability condition there holds

$$\int_{\Delta_j} p(x)\,dx \leqq \frac{C_1}{n};$$

(5) $|\Delta_j| \leqq \frac{C_2}{n}$, where $|\Delta_j|$ is the length of Δ_j — the uniformly small geometrical sizes condition.

C_j for $j=1, 2$ in (4) and (5) are an absolute constants.

First, Dupach (see [6, p. 140]) proved the following

Theorem 1. *Let $f(x)$ have continuous derivative f' on $[a, b]$ and satisfy conditions* (3), (4), (5). *Then*

$$r_n = O(n^{-3/2}).$$

Therefore, the Monte-Carlo method constructed above has an overconvergent probable error.

Let $f \in L_p[a, b]$ for $p \geqq 1$.

$$\omega(f, h)_{L_p[a,b]} = \sup\{\{\int_a^{b-t} |f(x+t)-f(x)|^p\,dx\}^{1/p} : 0 \leqq t \leqq h\}.$$

We shall use the following modulus from [3]:

$$\tau(f, t)_{p'p[a,b]} = \|\omega(f, \cdot\,; t)_{p'[a,b]}\|_{p[a,b]},$$

where

$$\omega(f, x; \delta)_{p'[b,a]} = \{\frac{1}{2\delta}\int_{-\delta}^{\delta} |f(x+t)-f(x)|^{p'}\,dt\}^{1/p'}$$

and $|f(x+t)-f(x)|$ is defined as 0 if x or $x+t$ are not in $[a, b]$, and $p' \geqq 1$. For $p' \leqq p$ in [3] was proved that

$$\tau(f, h)_{p'p[a,b]} \leqq \omega(f, h)_{L_p[a,b]} \leqq 2\tau(f, h)_{p'p[a,b]}, \tag{6}$$

$$\omega(f, x; \delta) = \sup\{|f(x+t)-f(x)| : t, t+h \in [x-\frac{\delta}{2}, x+\frac{\delta}{2}] \cap [a, b]\}.$$

We define the average modulus of smoothness as follows

$$\tau(f, \delta)_{p[a,b]} \leqq \|\omega(f, ., \delta)\|_{p[a,b]}.$$

Evidently,

$$\omega(f, h)_{L_p[a,b]} \leqq \tau(f, \delta)_{p[a,b]}. \tag{7}$$

Definition 2. Let $L(x)$ be a positive measurable function on the interval $(0, \infty)$. $L(x)$ is a slowly varying function if and only if for any $C>0$ holds

$$\lim_{x\to\infty} \frac{L(Cx)}{L(x)} = 1.$$

If $f(x)$ is a slowly varying function, for any $\varepsilon>0$ holds

$$\lim_{x\to\infty} f(x)x^{\varepsilon} = \infty, \text{ and } \lim_{x\to\infty} f(x)x^{-\varepsilon} = 0 \text{ (see [5, p. 24]).} \tag{8}$$

I. I. Dimov and O. I. Tonev [2] improved Dupach's result in the following way.

Theorem 2. *Let* $f(x) \in C[a, b]$, *and satisfies condition* (3), (4), (5). *Then*

$$r_n = O(\tau(f, n^{-1})_{2[a,b]} n^{-1/2}).$$

In this article we shall prove the following

Theorem 3. *Let* $f(x) \in L_2[a, b]$ *and satisfy conditions* (3), (4), (5). *Then*

$$r_n = O(\omega(f, n^{-1})_{L_2[a,b]} n^{-1/2}). \tag{9}$$

Remark 1. If $\mathrm{Var}(\theta_n) = \infty$, the normal convergence holds if and only if when $u(\tau) = \int_a^b f(x)\chi(|f(x)| \leqq \tau)\,dx$ is slowly varying function. In this case we have that $r_n = O(n^{-1/2+\varepsilon})$, where ε is arbitrary positive number (see (8)), therefore the above Monte-Carlo method has not overconvergent probable error.

Remark 2. From properties of $\omega(f, h)_{L_2[a,b]}$ (see [4, p. 116]), and from [1] it follows that the order of the estimate (4) is the best possible. The Dimov and Tonev result follows from the Theorem 3 and (7).

2. Auxiliary result

We define

(10) $$f_n(x)=n\int_{\frac{j}{n}}^{\frac{j+1}{n}} f(u)\,du \text{ for } x\in[j/n,\ (j+1)/n], \text{ and}$$

$$j=0,\ 1,\dots,n-1, \text{ and } f(1-1/n)=f(1).$$

We shall prove the following

Lemma 1. *Let* $f\in L_{p[0,1]}$ *then*

$$\sum_{j=0}^{n-1}\|f-f_n\|_{L_p[j/n,(j+1)/n]}\leqq\tau(f,\ h)_{1p[0,1]}.$$

P r o o f. Using the Minkovski inequality and (10) we obtain

$$\sum_{j=0}^{n-1}\|f-f_n\|_{L_p[j/n,(j+1)/n]}$$

$$\leqq\sum_{j=0}^{n-1}\left\{\int_{\frac{j}{n}}^{\frac{j+1}{n}}|f(x)-n\int_{\frac{j}{n}}^{\frac{j+1}{n}}f(u)\,du|^p\,dx\right\}^{1/p}$$

$$\leqq\sum_{j=0}^{n-1}\left\{\int_{\frac{j}{n}}^{\frac{j+1}{n}}n\int_{\frac{j}{n}}^{\frac{j+1}{n}}|f(x)-f(u)|^p\,du\right\}^{1/p}dx$$

$$\leqq\sum_{j=0}^{n-1}2\left\{\int_{\frac{j}{n}}^{\frac{j+1}{n}}\omega(f,\ x;\ t)^p_{1[\Delta_j]}\,dx\right\}^{1/p}$$

$$\leqq\sum_{j=0}^{n-1}2\left\{\int_{\frac{j}{n}}^{\frac{j+1}{n}}\omega(f,\ x;\ t)^p_{1[a,b]}\,dx\right\}^{1/p}\leqq\tau(f,\ h)_{1p[a,b]}.\ ■$$

3. Proof of Theorem 3

Without loss of generality we can suppose that $[a,\ b]\equiv[0,\ 1]$ and $q(x)\equiv 1$. After that we set $\Delta_j=[j/n,\ (j+1)/n]$, for $j=0,\ 1,\dots,n-1$. It is easy to see that the conditions (3), (4), (5) follow from the above assumptions. Now we consider the function $f(\xi_{nk})$ of the random variable ξ_{nk}, where ξ_{nk} is a random point uniformly distributed on Δ_k. Then $I=\sum_{k=0}^{n-1}I_k$, where $I_k=\int_{\Delta_k}f(x)\,dx$. For θ_n we obtain

$\theta_n=\frac{1}{n}\sum_{k=0}^{n-1} f(\xi_{nk})$. For the mathematical expectation of $f(\xi_{nk})$ we get $Mf(\xi_{nk})$ $=f_n((k-1)/n)$ for $k=0, 2,\dots,n-1$. Applying Lemma 1 for $p=2$ we get

$$\mathrm{Var}(\theta_n)=\mathrm{Var}\Big(\frac{1}{n}\sum_{k=0}^{n-1} f(\xi_{nk})\Big)=n^{-2}\sum_{k=0}^{n-1}\mathrm{Var}(f(\xi_{nk}))$$

$$=\frac{2}{n}\sum_{k=0}^{n-1}\tau(f, n^{-1})_{L_2[\Delta_j]}\leqq\frac{2}{n}\tau(f, n^{-1})_{L_2[0,1]}.$$

Now Theorem 3 follows immediately from (6) for $p=2$ and the Tchebycheff inequality. ∎

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