Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica
Balkanica

New Series Vol. 6, 1992, Fasc. 3

Some Fixed Point Theorems on an Arbitrary Metric Space

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Presented by Bl. Sendov

In this note some fixed point theorems, in an arbitrary metric space, has been presented which extend the theorems of G. E. Hardy and T. D. Rogers, [2], and L. B. Ciric [5].

1. Introduction

Let S be a set and T be a mapping from S to S. An element $x \in S$ is called a fixed point of T if

$$Tx = x.$$

The theory of fixed points began in 1912 by L. J. Brouwer, who proved that any continuous mapping of the closed unit ball in $\mathbb{R}^n$ into itself has a fixed point and was followed in 1922 by S. Banach's contraction principle, which states that any mapping T of a complete metric space X into itself that satisfies for some $0 < \alpha < 1$, the inequality:

$$(1) \qquad d(Tx, Ty) \leqq \alpha d(x, y),$$

for all $x, y \in X$, has a unique fixed point [1].

A number of generalizations of this result have studied by various authors, [4], [5], [6], [7]. In 1971 S. Reich proved a fixed point theorem for mapping, T, from a complete metric space into itself which instead of the contraction property, satisfying the condition

$$(2) \qquad d(Tx, Ty) \leqq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty)$$

$\forall x, y \in X$, $a_1 + a_2 + a_3 < 1$, $(a_1 \geqq 0,\ a_2 \geqq 0,\ a_3 \geqq 0)$, [3]. In 1973 G. E. Hardy and T. D. Rogers generalized these results to continuous mapping T of a complete metric space X into itself satisfying

$$(3) \quad d(Tx, Ty) \leqq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)$$

for all $x, y \in X$ where $a_i \geqq 0$ and $\sum_{i=1}^{5} a_i < 1$, [2].

To Prof. Bl. Sendov on his 60th birthday

2. Main results

In this section we prove two fixed point theorems for a mapping, T, from an arbitrary metric space into itself satisfying some special conditions.

Theorem 1. *If T is a mapping from an arbitrary metric space X into itself such that*

(a) $d(Tx, Ty) < a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)$,

where $a_i \geqq 0$, $0 < \sum_{i=1}^{5} a_i \leqq 1$ holds $\forall x, y \in X, (x \neq y)$.

(b) *There exists a point $u \in X$ such that*

$$f(u) = \mathrm{Inf}\,\{f(x) : f(x) = d(x, Tx),\ x \in X\},$$

then there exists a unique fixed point of T in X.

Proof. Let $u \neq Tu$, otherwise u is a fixed point of T. Put $x = u$ and $y = Tu$ in (a). Then we have

$d(Tu, T^2u) < a_1 d(u, Tu) + a_2 d(u, Tu) + a_3 d(Tu, T^2u) + a_4 d(u, T^2u) + a_5 d(Tu, Tu)$

$\Rightarrow d(Tu, T^2u) < a_1 d(u, Tu) + a_2 d(u, Tu) + a_3 d(Tu, T^2u) + a_4 [d(u, Tu) + d(Tu, T^2u)]$.

So, we get

$$(4) \qquad (1 - a_3 - a_4) d(Tu, T^2u) < (a_1 + a_2 + a_4) d(u, Tu).$$

Similarly, putting $x = Tu$ and $y = u$ in (a), we obtain

$d(T^2u, Tu) < a_1 d(Tu, u) + a_2 d(Tu, T^2u) + a_3 d(u, Tu) + a_4 d(Tu, Tu) + a_5 d(u, T^2u)$

$\Rightarrow d(T^2u, Tu) < a_1 d(Tu, u) + a_2 d(Tu, T^2u) + a_3 d(u, Tu) + a_5 [d(u, Tu) + d(Tu, T^2u)]$.

So, we get

$$(5) \qquad (1 - a_2 - a_5) d(T^2u, Tu) < (a_1 + a_3 + a_5) d(u, Tu).$$

From (4) and (5), we have

$$(2 - a_2 - a_3 - a_4 - a_5) d(Tu, T^2u) < (2a_1 + a_2 + a_3 + a_4 + a_5) d(u, Tu).$$

But, since $a_1 + a_2 + a_3 + a_4 + a_5 \leqq 1$, we have

$$2a_1 + a_2 + a_3 + a_4 + a_5 \leqq 2 - a_2 - a_3 - a_4 - a_5.$$

So, we have

$$d(Tu, T^2u) < \frac{(2a_1 + a_2 + a_3 + a_4 + a_5)}{(2 - a_2 - a_3 - a_4 - a_5)} d(u, Tu)$$

and $d(Tu, T^2u) < d(u, Tu)$.

Now, from (b), we assume that $f(u)=\text{Inf}\ \{f(x) : f(x)=d(x, Tx), x\in X\}$ for this $u\in X$, then we get

$$f(Tu)<f(u)=\text{Inf}\ \{f(x) : x\in X\}$$

which is a contradiction.

So, $u\neq Tu$ does not hold and therefore $Tu=u$. Thus u is a fixed point of T.

Uniqueness of the fixed point of T

Let v be another fixed point of T. Namely, assume that $Tv=v$. Then, we have

$$d(u, v)=d(Tu, Tv)<a_1 d(u, v)+a_2 d(u, Tu)+a_3 d(v, Tv)+a_4 d(u, Tv)+a_5 d(v, Tu)$$

$$=a_1 d(u, v)+a_4 d(u, v)+a_5 d(v, u)=(a_1+a_4+a_5)d(u, v).$$

But, since $a_1+a_4+a_5<1$, this is a contradiction.

Remark. 1) For a compact metric space X and continuity of T, condition (b) always holds but the converse is not always true.

2) If we take $a_4=0$ and $a_5=0$ on the theorem, we obtain the result of S. Reich [3] in an arbitrary metric space.

3) If we take $a_2=a_3=a_4=a_5=0$ on the theorem, we obtain non-complete metric space generalization of S. Banach's theorem with condition (b).

Now, let us give an example for our theorem which does not satisfy the conditions of G. E. Hardy, T. D. Rogers [2].

Example. Let $X=(0, \frac{1}{2}]$ with the metric $d(x, y)=|x-y|$. Define the mapping $T: X\to X$

$$T(x)=\sqrt{\frac{1-x}{2}}.$$

Then, $\forall x, y\in(0, \frac{1}{2}]$ we have

$$d(Tx, Ty)=|Tx-Ty|=\left|\sqrt{\frac{1-x}{2}}-\sqrt{\frac{1-y}{2}}\right|=\frac{1}{\sqrt{2}}\left|\sqrt{1-x}-\sqrt{1-y}\right|$$

$$=\frac{1}{\sqrt{2}}\left|\frac{x-y}{\sqrt{1-x}+\sqrt{1-y}}\right|<\frac{1}{\sqrt{2}}|x-y|, \text{ (because } 0<x\leqq\frac{1}{2},\ 0<y\leqq\frac{1}{2} \text{ and } x\neq y).$$

So, we can find the numbers $a_i\geqq 0$, with $0<\sum_{i=1}^{5} a_i\leqq 1$, such that

$$d(Tx, Ty)<a_1 d(x, y)+a_2 d(x, Tx)+a_3 d(y, Ty)+a_4 d(x, Ty)+a_5 d(y, Tx).$$

On the other hand, we have

$$d(x, Tx)=|x-Tx|=|x-\sqrt{\frac{1-x}{2}}|.$$

This attains its minimum at $d(x, Tx)=0$.

$$\text{So, } |x-\sqrt{\frac{1-x}{2}}|=0 \leftrightarrow x-\sqrt{\frac{1-x}{2}}=0$$

$$\leftrightarrow x^2=\frac{1-x}{2}.$$

$$\leftrightarrow x_{1,2}=-1, \frac{1}{2}.$$

But, since $-1\notin(0, \frac{1}{2}]$, function attains its infimum at $x=\frac{1}{2}$ and this point is also the fixed point of T.

On the other hand, since $(0, \frac{1}{2}]$ is not complete, then the theorem of G. E. Hardy and T. D. Rogers [2] can not be used in this space.

L. B. Ciric (See [5]) defines a mapping T on a metric space X into itself to be a quasi-contraction if and only if there exists a number c, with $0\leqq c<1$, such that

$$d(Tx, Ty)\leqq c.\operatorname{Max}\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all x, y in X. He defines X to be T-orbitally complete if and only if every Cauchy sequence which is contained in the sequence $\{x, Tx, \ldots, T^n x, \ldots\}$ for some x in X converges in X.

He then proves the following theorem:

"Let T be a quasi-contraction on a metric space X into itself and let X be T-orbitally complete. Then T has a unique fixed point in X".

Now, we prove a similar theorem without any condition on X.

Theorem 2. *If T is a mapping from an arbitrary metric space X into itself such that*

a) *There exists a number k, with $0<k<\frac{1}{2}$, such that*

$$d(Tx, Ty)<k.\operatorname{Max}\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all x, y in $X, x\neq y$.

b) *There exists a point $u\in X$ such that*

$$f(u)=\operatorname{Inf}\{f(x) : f(x)=d(x, Tx), x\in X\},$$

then there exists a unique fixed point of T in X.

Proof. Let $u\neq Tu$, otherwise u is a fixed point of T. Put $x=u$ and $y=Tu$ in (a), then we have

$$d(Tu,\ T^2u) < k\,.\,\mathrm{Max}\,\{d(u,\ Tu),\ d(u,\ Tu),\ d(Tu,\ T^2u),\ d(u,\ Tu),\ d(Tu,\ Tu)\}$$
$$= k\,.\,\mathrm{Max}\,\{d(u,\ Tu),\ d(u,\ T^2u),\ d(Tu,\ T^2u)\}$$

But, we may omit the term $d(Tu,\ T^2u)$ in the right hand part of the previous inequality (otherwise it gives a contradiction). So we get

$$d(Tu,\ T^2u) < k\,.\,\mathrm{Max}\,\{d(u,\ Tu),\ d(u,\ T^2u)\}$$
$$\leqq k\,.\,\mathrm{Max}\,\{d(u,\ Tu),\ d(u,\ Tu)+d(Tu,\ T^2u)\} = k\,[d(u,\ Tu)+d(Tu,\ T^2u)].$$

Therefore, we obtain

$$(1-k)d(Tu,\ T^2u) < kd(u,\ Tu) \text{ or}$$
$$d(Tu,\ T^2u) \leqq \frac{k}{1-k}d(u,\ Tu).$$

But since $k<\frac{1}{2}$ or $\frac{k}{1-k}<1$, we have

$$d(Tu,\ T^2u) < d(u,\ Tu).$$

On the other hand, we know from (b) that

$$f(u) = \mathrm{Inf}\,\{f(x) : f(x) = d(x,\ Tx),\ x \in X\}$$

for this u. Then, we obtain

$$f(Tu) < f(u).$$

But this contradicts to the definition of $f(u)$. Uniqueness can be seen easily from (a).

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Received 16.01.1992