Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica
Balkanica

New Series Vol. 6, 1992, Fasc. 4

Some Sequence Spaces with an Index

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The object of this note is to extend the concept $\hat{BV}$ to V_p.

1. Introduction

We write l_∞ and c, respectively, for the Banach spaces of bounded and convergent sequences normed as usual by $\|x\| = \sup_n |x_n|$. (Throughout x is written for a sequence $\{x_n\}$ of complex numbers.) We write D for the shift operator, that is,

$$D((x_n)) = (x_{n+1}).$$

We recall (see, S. Banach [1]) that a Banach limit L is defined as a nonnegative linear functional on l_∞ such that L is invariant under the shift operator (that is, $L(Dx) = L(x)$ for all $x \in l_\infty$) and such that $L(e) = 1$, where $e = (1, 1, \ldots)$. Various types of limits, including Banach limits, are considered by G. Das [2]. A sequence $x \in l_\infty$ is said to be almost convergent to the value s (see, G. G. Lorentz [5]) if $L(x) = s$ for all Banach limits L, that is, all Banach limits coincide. Let $\hat{c}$ denote the set of all almost convergent sequences.

It is also natural to accept that almost convergence must be related to some concept $\hat{BV}$ in the sme vein as convergence is related to the concept of BV. BV denotes the set of all sequences of bounded variation and a sequence in $\hat{BV}$ will mean a sequence of almost bounded variation. A new sequence spaces $\hat{BV}$ was introduced by S. Nanda and K. C. Nayak [6].

The main object of this note is to extend this concept and define the set of almost bounded variation sequences with index $p \geqq 1$.

2. Definition

For any sequence x, write

(2.1) $$d_{mn}=d_{mn}(x)=\frac{1}{m+1}\sum_{i=0}^{m} x_{n+1}$$

G. G. Lorentz [5] proved that

$$\hat{c}=\{x: \lim_{m\to\infty} d_{mn}(x) \text{ exists uniformly in } n\}.$$

We now extend the definition of d_{mn} to $m=-1$ by taking

(2.2) $$d_{-1,n}=x_{n-1}.$$

We then write for $m,\ n\geqq 0$,

(2.3) $$t_{mn}=t_{mn}(x)=d_{mn}(x)-d_{m-1,n}(x).$$

A straightforward calculation then shows that

(2.4) $$t_{mn}=\frac{1}{m(m+1)}\sum_{v=1}^{m} v(x_{n+v}-x_{n+v-1}).$$

We define (see, S. Nanda and K. C. Nayak [6])

$$\widehat{BV}=\{x: \sum_{m} |t_{mn}| \text{ converges uniformly in } n\}$$

and

$$\hat{\hat{BV}}=\{x: \sup_{n} \sum_{m} |t_{mn}|<\infty\}.$$

(Here and afterwards summation without limits runs from 0 to ∞).

We now extend the definition of $\widehat{BV}$ to a more general space V_p, where V_p denotes the set of all sequences of almost bounded variation with index $p\geqq 1$.

We now write,

$$V_p=\{x: \sum_{m} m^{p-1}\ |t_{mn}|^p \text{ converges uniformly in } n\}$$

$$W_p=\{x: \sup_{n} \sum_{m} m^{p-1} |t_{mn}|^p<\infty\}.$$

In this case $p=1$, we write $\widehat{BV}$ and $\hat{\hat{BV}}$ in place of V_p and W_p respectively.

Theorem 1. $V_p\subset W_p$.

Proof. Suppose that $x\in V_p$. Hence there exists an integer $M>0$ such that

(2.5) $$\sum_{m\geqq M} m^{p-1}\ |t_{mn}|^p\leqq 1, \qquad \forall n.$$

It is enough to show that,

(2.6) $$\sum_{m=1}^{M-1} m^{p-1}\ |t_{mn}|^p=O(1), \qquad \forall n.$$

(2.5) implies that, for $m \geqq M$

$$|t_{mn}|^p \leqq \frac{1}{m^{p-1}} \leqq 1 \tag{2.7}$$

and so this implies

$$|t_{mn}| \leqq 1 \qquad (m \geqq M, \ \forall n). \tag{2.8}$$

Since

$$x_{n+m} - x_{n+m-1} = (m+1)\, t_{mn} - (m-1)_{m-1,n}. \tag{2.9}$$

It follows from (2.8) that for any fixed $m \geqq M$,

$$|x_{n+m} - x_{n+m-1}| = O(1) \qquad \forall n.$$

This implies that $|x_v - x_{v-1}|$ is bounded, and so $|t_{mn}|$ is bounded for all m and n. This completes the proof. ■

Before we consider the next inclusion relation, let first make a generalization of bounded variation. Write,

$$R_p = \{x\colon \sum_m m^{p-1}\, |x_m - x_{m-1}|^p < \infty, \ p \geqq 1\}. \tag{2.10}$$

Note that R_1 is the set of all sequences of bounded variation. We now prove

Theorem 2. $R_p \subset V_p$.

Proof. Let $x \in R_p$. By Hõlder's inequality for $p > 1$ and trivially for $p = 1$, we obtain,

$$|t_{mn}|^p \leqq \frac{1}{m(m+1)^p} \sum_{v=1}^{m} v^p\, |x_{n+v} - x_{n+v-1}|^p.$$

Hence,

$$\sum_{p=1}^{\infty} m^{p-1} |t_{mn}| \leqq \sum_{v=1}^{\infty} v^p\, |x_{n+v} - x_{n+v-1}| \sum_{m=v}^{\infty} \frac{1}{m^2}$$

$$\leqq \sum_{v=1}^{\infty} v^{p-1}\, |x_{n+v} - x_{n+v-1}|^p$$

$$\leqq \sum_{v=n}^{\infty} v^{p-1}\, |x_v - x_{v-1}|^p. \tag{2.11}$$

Since the uniform convergence of $\sum_m m^{p-1}\, |t_{mn}|^p$ follows at once from (2.11) and this completes the proof. ■

3. Topological Properties

Now from the sets V_p, W_p and R_p as follows we may define for $x \in V_p$ or W_p

(3.1) $$\|x\| = \sup\Big(\sum_{m=1}^{\infty} m^{p-1} |t_{mn}(x)|^p\Big)^{1/p}, \qquad p \geqq 1$$

and for $x \in R_p$

(3.2) $$\|x\|_R = \Big(\sum_m m^{p-1} |x_m - x_{m-1}|^p\Big)^{1/p} \qquad p \geqq 1.$$

We have

Theorem 3. *V_p, W_p and R_p are Banach spaces and*

(3.3) $$\|x\|_R \leqq \|x\|.$$

Proof. It is easy to ferify that all sets considered are normed linear spaces. We proceed to prove their completeness.

Let (x^i) be a Cauchy sequence in V_p. Then $(x_n^i)_{i=0}^{\infty}$ is a Cauchy sequence in C for each n. Therefore $x_n^i \to x_n$ (say). We now show that $x \in V_p$ and $\|x^i - x\| \to 0$. Since (x^i) is a Cauchy sequence in V_p, given $\varepsilon > 0$, there exists N such that for $i, j > N$,

$$\sum_m m^{p-1} |t_{mn}(x^i - x^j)| < \varepsilon \qquad (\forall n).$$

Therefore for any M and $i, j > N$

$$\sum_{m=0}^{M} m^{p-1} |t_{mn}(x^i - x^j)| < \varepsilon \qquad (\forall n).$$

Now taking limit as $j \to \infty$ and as $M \to \infty$ we get for $i > N$,

(3.4) $$\sum_m m^{p-1} |t_{mn}(x^i - x)| \leqq \varepsilon \qquad (\forall n).$$

Now taking the supremum with respect to n, we obtain

$$\|x^i - x\| \leqq \varepsilon \qquad (i > N).$$

Hence $x^i \to x$ and $x \in V_p$.

This proves that V_p is complete. The proof of the fact that W_p and R_p are complete can be done in a similar way. The relation (3.3) is evident from the inequality (2.11).

This completes the proof. ∎

Acknowledgement

The author wishes to thank Prof. G. Das, Utkal University, Orissa, for his valuable suggestions.

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Received 14. 02. 1992