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Asymptotic Bifurcation in Systems for Control with Delay

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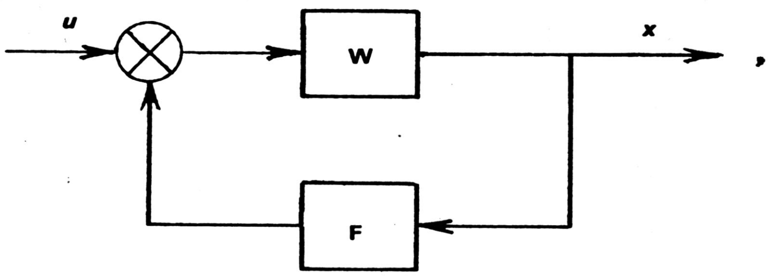
In this paper periodic states in systems for control with delay are considered. In the problem for forsed periodic oscillations sufficient conditions are obtained for existence of regular asymptotic bifurcation points.

1. Introduction

In [1] scalar feedback loops with a single nonlinearity are investigated which correspond to the following block diagram



where W is a linear stationary (time-invariance) element, F is a nonlinear element and u is a control signal.

This paper deals with the problem for asymptotic bifurcation of the forced periodic oscillations in the system for control represented by the above block diagram. Forced T-periodic oscillation is said to be a T-periodic state x conditioned by the T-periodic control signal u and by the T-periodicity of F.

The main aim of the present work is to apply some classical theorems of nonlinear analysis to our problem and to generalize the results obtained for systems without delay in [2—3]. The proofs we propose are based on the theory of completely continuous vector fieds which is related to the theory of topological degree (see [4-5]).

2. Statement of the problem

Denote integrals on the whole real axis by integral sing without boundaries. Further, we assume that:

A. W is a linear stationary element whose input-output relation ($u \to x$) satisfies the equation

$$(1)\quad x^{(n)}(t)+\sum_{i=0}^{i=n-1}\int x^{(i)}(t-s)\,\mathrm{d}\gamma_i(s;\ \lambda)=\sum_{i=0}^{i=n-1}\int u^{(i)}(t-s)\,\mathrm{d}\beta_i(s;\ \lambda),$$

where $\gamma_i(\cdot;\lambda)$, $\beta_i(\cdot;\lambda)$, $0\leqq i\leqq n-1$, are real-valued measures in $\mathbf{R}^1$ with supp $\gamma_i(\cdot;\lambda)\subseteq[0,r]$, supp $\beta_i(\cdot;\lambda)\subseteq[0,r]$, $0\leqq i\leqq n-1$, $r\geqq 0$, $n\geqq 1$;

B. F is a nonlinear element with characteristic function $f(t, y_1, y_2, \ldots, y_l;\ \lambda)$ which is T-periodic with respect to the time variable and with stationary delays in the phase variables, i. e. the input-output relation ($x \to w$) satisfies the equation

$$w(t)=f\left(t,\ \int x(t-s)\,\mathrm{d}\alpha_1(s;\ \lambda),\ldots,\ \int x(t-s)\,\mathrm{d}\alpha_l(s;\ \lambda);\ \lambda\right),$$

where $\alpha_i(\cdot;\lambda)$, $1\leqq i\leqq l$, are real-valued measures in $\mathbf{R}^1$ with supp $\alpha_i(\cdot;\lambda)\subseteq[0,r]$, $1\leqq i\leqq l$;

C. At each fixed $v\in\mathbf{C}(T)$ the formulas

$$\int v(t-s)\,\mathrm{d}\gamma_i(s;\ \lambda),\quad \int v(t-s)\,\mathrm{d}\beta_i(s;\ \lambda),\quad 0\leqq i\leqq n-1,$$

$$\int v(t-s)\,\mathrm{d}\alpha_i(s;\ \lambda),\quad 1\leqq i\leqq l,$$

define continuous, with respect to λ, mappings from Δ into $\mathbf{C}(T)$ where $\Delta=[\lambda', \lambda'']\subset\mathbf{R}^1$ and $\mathbf{C}(T)$ is the Banch space of the continuous real-valued T-periodic functions provided with the usual norm

$$\|x\|=\max_{0\leq t\leq T}|x(t)|.$$

Introducing the notations

$$L(p;\ \lambda)=p^n+\sum_{i=0}^{i=n-1}p^i\int\exp(-ps)\,\mathrm{d}\gamma_i(s;\ \lambda),$$

$$M(p;\ \lambda)=\sum_{i=0}^{i=n-1}p^i\int\exp(-ps)\,\mathrm{d}\beta_i(s;\ \lambda),$$

we obtain that $W(p;\ \lambda)=M(p;\ \lambda)/L(p;\ \lambda)$ is the transfer function of W. As in [1] we have that the equation

$$(2)\qquad x(t)=W(p;\ \lambda)\,F(t,\ x_t;\ \lambda),$$

describes the dynamic of the system for control where

$$F(t,\ x_t;\ \lambda)=u(t)+f(t,\ \Psi_1(\lambda)\,x_t,\ \Psi_2(\lambda)\,x_t,\ldots,\ \Psi_l(\lambda)\,x_t;\ \lambda),$$

with

$$\Psi_i(\lambda)\, x_t = \int x(t-s)\, d\alpha_i(s;\ \lambda), \qquad 1 \leq i \leq l.$$

Further, we assume that the following conditions hold

$$L(l_k^T;\ \lambda) \neq 0,\ k \in \mathbf{Z},\ \lambda \in \Delta,$$

where $l_k^T = 2\pi k\sqrt{-1}\,/\,T$.

This assumption implies that the T-periodic problem for equation (1) is solvable with a solution of the form

$$x(t) = \int_0^T G(t-s,\ T;\ \lambda)\, u(s)\, ds \overset{\text{def}}{=} \mathfrak{G}(T;\ \lambda)\, u,\ u \in \mathbf{C}(T),$$

where $G(t,\ T;\ \lambda)$ is a T-periodic function of bounded variation in $[0,\ T]$ with a Fourier expansion

$$G(t,\ T;\ \lambda) = T^{-1} \sum_{k \in \mathbf{Z}} W(l_k^T;\ \lambda) \exp(l_k^T t).$$

It is clear that the set of the numbers $|k\, W(l_k^T;\ \lambda)|$ $k \in \mathbf{Z}$, is bounded which gives that $\mathfrak{G}(T;\ \lambda)$ is a compact operator in $\mathbf{C}(T)$.

The problem for forced T-periodic oscillations in the system for control is equivalent to the problem for T-periodic solutions of equation (2) which in the case considered can be written as

$$x = \mathfrak{G}(T;\ \lambda)\, F(t,\ x_t;\ \lambda) \overset{\text{def}}{=} \mathfrak{P}(x,\ T;\ \lambda). \tag{3}$$

Obviously, if $u \in \mathbf{C}(T)$ and $f \in \mathbf{C}(R^{1+l} \times \Delta,\ R^1)$ then $\mathfrak{P}(\cdot;\ T;\ \lambda)$ is a completely continuous operator in $\mathbf{C}(T)$.

In the next sections we consider the problem for bifurcation of the solutions of equation (3) with respect to λ.

3. Forced T-periodic oscillations

In what follows, we assume that $f \in \mathbf{C}(R^{1+l} \times \Delta,\ R^1)$ and that f is an asymptotic linear function in $\mathbf{R}^l$, i. e.

$$\lim_{\Sigma |y_i| \to \infty} \sup_t \left(\sum_{i=1}^{i=l} |y_i| \right)^{-1} |f(t,\ y_1,\ y_2, \ldots,\ y_l;\ \lambda) - \sum_{i=1}^{i=l} a_i(\lambda)\, y_i| = 0, \tag{4}$$

where $a_i(\lambda)$, $1 \leq i \leq l$, are continuous functions in Δ.

From (4) it follows that $\mathfrak{P}(\cdot,\ T;\ \lambda)$ is an asymptotic linear operator, i. e.

$$\lim_{\|x\| \to \infty} \|x\|^{-1} \|\mathfrak{P}(x,\ T;\ \lambda) - \mathfrak{P}'(\infty,\ T;\ \lambda)\, x\| = 0,\ \ x \in \mathbf{C}(T),$$

where the asymptotic derivative has the form

$$\mathfrak{P}'(\infty,\ T;\ \lambda)\,x(t)=\mathfrak{G}\,(T;\ \lambda)\left(\sum_{i=1}^{i=l} a_i(\lambda)\,\Psi_i(\lambda)\,x_t\right).$$

The compactness of $\mathfrak{G}(T;\lambda)$ implies the compactness of $\mathfrak{P}'(\infty, T;\ \lambda)$ in $C(T)$. The Fourier expansion

$$v(t)\overset{\text{a.e.}}{=}\sum_{k\in Z} v_k g_k(t,\ T),\ g_k(t,\ T)=\exp(l_k^T t)/\sqrt{T},$$

holds for every $v\in L_2(0,\ T)$ and allows us to find another representation for the asymptotic derivative

$$\mathfrak{P}'(\infty,\ T;\ \lambda)\,v=\sum_{k\in Z}\mu_k(T;\ \lambda)\,v_k g_k(t,\ T),\ v\in C(T),$$

where

$$\mu_k(T;\ \lambda)=W(l_k^T;\ \lambda)\left(\sum_{i=1}^{i=l} a_i(\lambda)\int\exp(-l_k^T s)\,d\alpha_i(s;\ \lambda)\right).$$

This representation is the natural extension of $\mathfrak{P}'(\infty,\ T;\ \lambda)$ on the space $L_2(0,\ T)$. It is clear that $\mathfrak{P}'(\infty,\ T;\ \lambda)$ is a normal operator, besides, the relation

$$\lim_{|k|\to\infty}|\mu_k(T;\ \lambda)|=0,$$

implies the compactness of the same operator in $L_2(0,\ T)$. The conditions of the theorem we shall prove in this section exclude the case when $\mathfrak{P}'(\infty,\ T;\ \lambda)\equiv 0$. In this way we obtain that the operator $\mathfrak{P}'(\infty,\ T;\ \lambda)$ has entirely point spectrum in $L_2(0,\ T)$ with eigenvalues $\{\mu_k(T;\ \lambda),\ k\in Z\}$ and corresponding eigenfunctions $\{g_k(t,\ T),\ k\in Z\}$. Denote $SP=\{\mu_k(T;\ \lambda),\ k\in Z\}$.

The set of the eigenvalues of $\mathfrak{P}'(\infty,\ T;\ \lambda)$ in $C(T)$ consists of the real numbers which belong to SP. It is not difficult to see that for every real $\mu\in SP$ the corresponding rootspace $H(\mu)\subset C(T)$ coincides with the real finite-dimensional spase generated by the eigenfunctions $g_k(t,\ T)$ for which $\mu_k(T;\ \lambda)=\mu$.

Therefore $H(\mu)$ has the property

$$\dim H(\mu)\equiv\begin{cases}1\ (\mathrm{mod}\ 2), & \mu=\mu_0(T;\ \lambda)\\ 0\ (\mathrm{mod}\ 2), & \mu\neq\mu_0(T;\ \lambda)\end{cases}. \tag{5}$$

It is clear that the eigenvalues depend continuously on λ.

Definition 1. [4] The number $\lambda_*\in\Delta$ is said to be a regular asymptotic bifurcation point for the solutions of equation (3) if for an arbitrary $\varepsilon>0$ there is a constant $R(\varepsilon)$ sich that for every $R\geqq R(\varepsilon)$ there exists $\lambda(R,\ \varepsilon)\in\Delta$ for which $|\lambda_*-\lambda(R,\ \varepsilon)|<\varepsilon$ and the equation $x=\mathfrak{P}(x,\ T;\ \lambda(R,\ \varepsilon))$ has a solution $x[\lambda(R,\ \varepsilon)]\in C(T)$ with $\|x[\lambda(R,\ \varepsilon)]\|=R$.

We define

$$\Phi(\lambda)=\mu_0(T;\lambda)-1=W(0;\lambda)\left(\sum_{i=1}^{i=l} a_i(\lambda)\int d\alpha_i(s;\lambda)\right)-1.$$

Theorem 1. *Let* $u\in\mathbf{C}(T)$ *and let* $\lambda_*\in(\lambda',\lambda'')$ *be such that the following conditions hold:*
(a) $\Phi(\lambda_*)=0$ *and* $\Phi(\lambda)$ *changes its sign at* λ_*;
(b) $\mu_k(T;\lambda)\neq 1$ *for* $k\in\mathbf{Z}\setminus(0)$ *and* λ *in a certain neighbourhood of* λ_*.
Then λ_* *is a regular asymptotic bifurcation point for the solutions of equation* (3).

Proof. Let $\mathfrak{J}$ be the identity operator in $\mathbf{C}(T)$. The conditions of the theorem imply that there is a neighbourhood Δ' of λ_*, $\Delta'\subset\Delta$, such that, if $\lambda\in\Delta'\setminus\{\lambda_*\}$ then the completely continuous vector field $\mathfrak{J}-\mathfrak{P}'(\infty,T;\lambda)$ is non-degenerate in $\mathbf{C}(T)$, i. e. $\mathfrak{P}'(\infty,T;\lambda)$ has no non-trivial fixed points in $\mathbf{C}(T)$.

From (a) it follows that for an arbitrary $\varepsilon>0$ there exist $\lambda_1,\lambda_2\in\Delta'$ such that

$$\lambda_*-\varepsilon<\lambda_1<\lambda_*<\lambda_2<\lambda_*+\varepsilon,$$

and

$$\Phi(\lambda_1)\Phi(\lambda_2)<0. \tag{6}$$

The absence of non-trivial fixed points of the asymptotic derivatives $\mathfrak{P}'(\infty,T;\lambda_i)$, $i=1,2$, implies that there is a constant $R(\lambda_1,\lambda_2)$ such that the relations

$$\Gamma(\mathfrak{J}-\mathfrak{P}(\cdot,T;\lambda_i),S_R)=\operatorname{ind}(0,\mathfrak{J}-\mathfrak{P}'(\infty,T;\lambda_i))=(-1)^{\rho(\lambda_i)},\ i=1,2, \tag{7}$$

are true for every R with $R\geqq R(\lambda_1,\lambda_2)$. Here, by $\Gamma(\cdot,\cdot)$ we denote rotation of vector fields (see [4]), $\rho(\lambda_i)$ is the sum of the algebraic multiplicities of the greater that 1 eigenvalues of $\mathfrak{P}'(\infty,T;\lambda_i)$, $i=1,2$, and $S_R=\{x\in\mathbf{C}(T):\|x\|<R\}$.

Further we suppose that $R\geqq R(\lambda_1,\lambda_2)$.

From (5) and (7) it follows

$$\Gamma(\mathfrak{J}-\mathfrak{P}(\cdot,T;\lambda_i),S_R)=\begin{cases}-1, & \Phi(\lambda_i)>0\\ 1, & \Phi(\lambda_i)<0\end{cases},\quad i=1,2. \tag{8}$$

Then (6) and (8) yield

$$\Gamma(\mathfrak{J}-\mathfrak{P}(\cdot,T;\lambda_1),S_R)\neq\Gamma(\mathfrak{J}-\mathfrak{P}(\cdot,T;\lambda_2),S_R). \tag{9}$$

It is not difficult to see that $\mathfrak{G}(T;\lambda)$ is uniformly continuous with respect to λ. Therefore

$$\{\bigcup_{x,\lambda}\mathfrak{P}(x,T;\lambda):\lambda\in[\lambda_1,\lambda_2],\ x\in M\},$$

is a compact set in $\mathbf{C}(T)$ for every bounded $M\subset\mathbf{C}(T)$.

This allows us to use the homotopic invariance of $\Gamma(\cdot,\cdot)$.

Now, if we assume that for every $\lambda \in [\lambda_1, \lambda_2]$ $\mathfrak{P}(\cdot, T; \lambda)$ has no fixed points in ∂S_R then the completely continuous vector fields $\mathfrak{I} - \mathfrak{P}(\cdot, T; \lambda_i)$, $i = 1, 2$, should be homotopic on ∂S_R which contradicts to (9).

Thus we obtain that for a certain $\lambda \in [\lambda_1, \lambda_2]$ there is a solution x of equation (3) with $\| x \| = R$. ∎

R e m a r k 1. In the case when (4) holds uniformly with respect to λ and $1 \notin SP$ for $\lambda \in \Delta$, one can prove that equation (3) has a solution for every $\lambda \in \Delta$ and the set of all these solutions is bounded in $\mathbf{C}(T)$, i. e. the problem for asymptotic bifurcation has no solution in Δ. It is clear that $1 \neq \mu_k(T; \lambda)$ for every sufficiently large $|k|$. Moreover, only the eigenvalue $\mu_0(T; \lambda)$ is certainly real number which justifies the conditions of theorem 1.

4. High frequency non-negative forced oscillations

In this section we assume additionally that the inequality

$$(10) \qquad f(t, y_1, y_2, \dots, y_l; \lambda) \geqq 0, \quad t \in \mathbf{R}^1, \ \lambda \in \Delta,$$

holds for $y_i \geqq 0$, $1 \leqq i \leqq l$, and

$$(11) \qquad L(0; \lambda) M(0; \lambda) > 0, \ \lambda \in \Delta.$$

Besides, we assume that the functions $\alpha_i(\cdot; \lambda)$, $1 \leqq i \leqq l$, $\lambda \in \Delta$, are non-decreasing.

It can be shown that (11) implies the existence of a constant T_0 such that the relations

$$L(l_k^T; \lambda) \neq 0, \ K \in \mathbf{Z}, \ \lambda \in \Delta,$$

$$G(t, T; \lambda) \geqq g > 0, \ t \in \mathbf{R}^1, \ \lambda \in \Delta,$$

are true for every T with $0 < T \leqq T_0$.

From (4) and (10) we obtain that $a_i(\lambda) \geqq 0$, $1 \leqq i \leqq l, \lambda \in \Delta$.

Let $K^+(T)$ denote the cone of the non-negative functions in $\mathbf{C}(T)$. Now it is clear that if $0 < T \leqq T_0$ and $u \in K^+(T)$ then $\mathfrak{P}(\cdot, T; \lambda)$ and $\mathfrak{P}'(\infty, T; \lambda)$ are positive operators with respect to the cone $K^+(T)$.

Further we assume that $0 < T \leqq T_0$.

Definition 2. The number $\lambda_* \in \Delta$ is said to be a regular asymptotic bifurcation point for the non-negative solutions of equation (3) if λ_* satisfies definition 1 and $x[\lambda(R, \varepsilon)] \in K^+(T)$.

Theorem 2. *Let $u \in K^+(T)$ and let $\lambda_* \in (\lambda', \lambda'')$ satisfies condition* (a) *of theorem* 1. *Then λ_* is a regular asymptotic bifurcation point for the non-negative solutions of equation* (3) .

P r o o f. Since the condition (a) of theorem 1 holds we obtain that there is a neighbourhood Δ' of λ_*, $\Delta' \subset \Delta$, such that if $\lambda \in \Delta'$ then $\mu_0(T; \lambda) > 0$, besides, if

$\lambda \neq \lambda_*$ then $\mu_0(T;\lambda) \neq 1$. In particular, $\mathfrak{P}'(\infty, T; \lambda)$, $\lambda \in \Delta'$, is a "strong" positive operator with respect to the cone $K^+(T)$ (see [6]).

Now, the equality

$$\mathfrak{P}'(\infty, T; \lambda)\, e(t) = \mu_0(T; \lambda)\, e(t), \quad e(t) \equiv 1,$$

implies that if $\lambda \in \Delta'$, then

$$r(\mathfrak{P}'(\infty, T; \lambda)) = \mu_0(T; \lambda),$$

besides, if $\lambda \neq \lambda_*$, then the positive vector field $\mathfrak{J} - \mathfrak{P}'(\infty, T; \lambda)$ is non-degenerate in $K^+(T)$. Here by $r(\cdot)$ we denote spectral radius.

Let $K^+_*(T)$ be the cone defined as follows

$$K^+_*(T) = \{x \in K^+(T) : \min_t x(t) \geqq g/G \max_t x(t)\},$$

where G satisfies the inequality

$$G(t, T; \lambda) \leqq G, \quad t \in \mathbf{R}^1, \quad \lambda \in \Delta'.$$

It is easy to see that $\mathfrak{G}(T; \lambda)\, K^+(T) \subset K^+_*(T)$. Then

$$\mathfrak{G}(T; \lambda)\, \mathfrak{F}\, K^+_*(T) \subset K^+_*(T).$$

for every operator $\mathfrak{F}$ which is positive with respect to $K^+(T)$.

Thus we obtain that $\mathfrak{P}(\cdot, T; \lambda)$ and $\mathfrak{P}'(\infty, T; \lambda)$, $\lambda \in \Delta'$, are positive operators with respect to the cone $K^+_*(T)$.

At this point we denote

$$S^+_R = \{x \in K^+_*(T) : \|x\| < R\}, \quad \partial S^+_R = \{x \in K^+_*(T) : \|x\| = R\}.$$

Let λ_1, λ_2 are chosen as in the proof of theorem 1. Then according to the general theory of the positive operators (see [4], [6]) we have

$$\operatorname{ind}(0, \mathfrak{P}'(\infty, T; \lambda_i); K^+_*(T)) = \begin{cases} 1, & \Phi(\lambda_i) < 0 \\ 0, & \Phi(\lambda_i) > 0 \end{cases}, \quad i = 1, 2.$$

The asymptotic linearity of $\mathfrak{P}(\cdot, T; \lambda_i)$, $i = 1, 2$, along with the latter conclusion imply that there is a constant $R(\lambda_1, \lambda_2)$ such that the relations

$$\text{(12)} \qquad \Gamma[\mathfrak{J} - \mathfrak{P}(\cdot, T; \lambda_1), S^+_R] = \operatorname{ind}(0, \mathfrak{P}'(\infty, T; \lambda_1); K^+_*(T))$$
$$\neq \operatorname{ind}(0, \mathfrak{P}'(\infty, T; \lambda_2); K^+_*(T)) = \Gamma[\mathfrak{J} - \mathfrak{P}(\cdot, T; \lambda_2), S^+_R],$$

are true for every R with $R \geqq R(\lambda_1, \lambda_2)$, where by $\Gamma[\cdot, \cdot]$ we denote rotation of positive fields.

Further we suppose that $R \geqq R(\lambda_1, \lambda_2)$.

If we assume that for every $\lambda \in [\lambda_1, \lambda_2]$ the completely continuous positive vector fields $\mathfrak{J} - \mathfrak{P}(\cdot, T; \lambda)$ are non-degenerate in ∂S^+_R then they should be positive homotopic on ∂S^+_R which contradicts to (12).

Hence there is $\lambda \in [\lambda_1, \lambda_2]$ such that equation (3) has a solution $x \in K^+_*(T)$ with $\|x\| = R$. ■

R e m a r k 2. In fact, in the proof of theorem 2 we showed that for an arbitrary M there is a value of the parameter for which equation (3) has a solution $x \in K^+(T)$ with

$$\min_t x(t) \geq M.$$

This corresponds more to the object of our problem and justifies the introducing of the cone $K_*^+(T)$.

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