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On Certain Transformations of Measurable Sets in R_N

K. C. Ray[+], *Kabita Mukhopadhyay*[++]

Presented by P. Kenderov

In this paper some properties of Lebesgue measurable sets in R_N (N-dimensional Euclidean space) have been studied under certain transformations as introduced by T. Neubrunn and T. Salat. Theorem 1 and Theorem 2 extend some results of M. Pal [3] and N. G. Saha and K. C. Ray [5] under less-restricted conditions.

Let Ω be a metric space and $\omega \in \Omega$. Then T. Neubrunn and T. Šalăt [2] introduced certain transformations T_ω of Lebesgue measurable sets into Lebesgue measurable sets of the real line under some conditions. Among other results M. Pal [3] in Theorem 1 extended Theorem 1.1 of [2] where the transformation T_ω transforms a measurable set in R_N (N-dimensional Euclidean space) into a measurable set in R_N. For this the transformations had to satisfy the three conditions (I), (II) and (III) as stated below, which are equivalent to the set of conditions as imposed by Neubrunn and Šalăt. But before stating these conditions we explain some notations:

(1) $|x|$ denotes the norm of the vector x in R_N. (2) $A \setminus B$ denotes the set of all those vectors of the set A which do not belong to the set B. (3) For $a \in R_N$, $A \subset R_N$ the symbol $\{|a-A|\}$ denotes the set of all numbers $|a-x|$ where $x \in A$. (4) $S[a, r]$ denotes a closed sphere in R_N with centre at a and radius r. (5) The Lebesgue measure of a measurable set A in R_N is denoted by $|A|$.

(I). There exists $\omega_0 \in \Omega$ such that for every sphere $K = S[a, r] \subset R_N$ and every sequence $\{\omega_n\}$ ($\omega_n \in \Omega$) converging to ω_0,

$$\lim_{n \to \infty} [\sup\{|a - T_{\omega_n}(K)|\} = r$$

holds.

(II). If E and F are measurable sets in R_N such that $F \subset E$, then for every $\omega \in \Omega$,

$$T_\omega(F) \subset T_\omega(E).$$

(III). If E be a measurable set in R_N and $\omega_n \to \omega_0$ (in Ω), then

$$\lim_{n \to \infty} |T_{\omega_n}(E)| = |T_{\omega_0}(E)| = |E|.$$

It appears that in condition (III) the transformation T_{ω_0} is measure-preserving. In this paper we see that this condition may be relaxed in the following way:

Let T be a non-singular linear transformation in R_N given by

$$T: x_i' = \sum_{j=1}^{N} a_{ij} x_j, \quad i=1, 2, \ldots, N$$

$$\text{where } \det T = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{vmatrix}$$

If $\Phi(x_{11}, x_{12}, \ldots, x_{1N}; \ \ldots; \ x_{N1}, x_{N2}, \ldots, x_{NN})$

$$= \begin{vmatrix} 1+x_{11} & x_{12} & \cdots & x_{1N} \\ x_{21} & 1+x_{22} & \cdots & x_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ x_{N1} & x_{N2} & \cdots & 1+x_{NN} \end{vmatrix}$$

then $\Phi(x_{11}, x_{12}, \ldots, x_{1N}; \ \ldots; \ x_{N1}, x_{N2}, \ldots, x_{NN})$ is a continuous function of N^2 variables and $\Phi(0,0,\ldots 0; \ \ldots; \ 0,0,\ldots,0)=1$. So, for arbitrary $\varepsilon(>0)$ there exists a $\delta'(>0)$ such that

$$\Phi(x_{11}, x_{12}, \ldots, x_{1N}; \ \ldots; \ x_{N1}, x_{N2}, \ldots, x_{NN}) > 1-\varepsilon$$

for $|x_{ij}|<\delta'$.

It, therefore, follows that corresponding to $1/2p^2$ (p being a positive integer) there exists a $\delta(>0)$ and a class of non-singular linear transformations J of the form

$$x_i' = \sum_{j=1}^{N} a_{ij} x_j, \quad i=1, 2, \ldots, N$$

for which $1-\delta<a_{ij}<1+\delta$ when $i=j$ and $-\delta<a_{ij}<\delta$ when $i \neq j$ ($i, j=1, 2, \ldots, N$) and so

$$|\det J| > 1 - \frac{1}{2p^2}.$$

Now if J is any non-singular linear transformation defined as above, we replace the condition (III) by the following:

(III)'. If E be a measurable set in R_N and $\omega_n \to \omega_0$ (in Ω), then

$$\lim_{n\to\infty} |T_{\omega_n}(E)| = |T_{\omega_n}(E)| = |J(E)$$

where J is a suitable non-singular linear transformation in R_N.

The transformation J as derived above will be applied to Theorem 1. But in Theorem 2 we need to change the positive number $1/2p^2$ (and so $\delta>0$) to choose a transformation J from a different class. However, this change will be evident from the context.

The result of Theorem 1 is a generalised form of Theorem 1 of [3] where the transformations T_ω satisfy the conditions (I), (II) and (III)'. Theorem 2 is an extension of Theorem 1 in which we prove the result contained in Theorem 1 of [5] under a set of less restricted conditions. In Theorem 3 we prove a result which may be compared to a theorem of S. Kurepa [1]. However, in doing so we follow the method of proof as adopted by K. C. Ray [4].

Theorem 1. *Let T_{ω_n} be the transformations satisfying the conditions (I), (II) and (III)' and $\omega_n \to \omega_0$ (in Ω). Let A be a set of positive measure in R_N and p be a positive integer. Then there exists a positive integer N_0 such that for $N_1, N_2, \ldots, N_p \geqq N_0$,*

$$A \cap T_{\omega_{N_1}}(A) \cap T_{\omega_{N_2}}(A) \cap T_{\omega_{N_p}}(A)$$

is a set of positive measure.

Proof. First, we observe that in view of [6] the expression in condition (III)' may be stated as follows:

$$\lim_{n\to\infty} |T_{\omega_n}(E)| = |T_{\omega_0}(E)| = |J(E)| = |\det J||E| = \alpha|E|$$

where $1-1/2p^2<\alpha<1+1/2p^2$.

Next, since A is a set of positive measure, there exists a sphere $K_1=S[a, r]$ such that $|K_1 \setminus A|<\varepsilon|K_1|$, where $0<\varepsilon<\frac{1}{2(2p^2+2p+1)}$. Let $K_2=S[a, s]$, where $s=(p/p+1)^{1/N_r}$ and $\sup\{|a-T_{\omega_n}(K_2)|\}=d_n$. By (I) there exists a positive integer N_1 such that for $n \geqq N_1$, $|d_n-s|<r-s$. So, for $n\geqq N_1$, $T_{\omega_n}(K_2\cap A) \subset K_1$. Also by (III)' there exists a positive integer N_2 sch that for $n\geqq N_2$, $||T_{\omega_n}(K_2\cap A)| -\alpha|K_2\cap A||<\frac{|K_2|}{4p^2}=\mu$, say where $1-\frac{1}{2p^2}<\alpha<1+\frac{1}{2p^2}$.

Again, let $N_0=\max(N_1, N_2)$ and $C=K_1\cap A$, $C_k=T_{\omega_{N_k}}(K_2\cap A)$ $(k=1, 2, \ldots, p)$. Since $C\cap C_1\cap\ldots\cap C_p=K_1\setminus[C'\cup C_1'\cup C_2'\cup\ldots\cup C_p'^k$ (dashes denote the complements with respect to K_1), we have for $N_1, N_2, \ldots, N_0 \geqslant N_0$

$$|C\cap C_1\cap C_2\cap\ldots\cap C_p| \geqq |K_1|-[|C'|+|C_1'|+\ldots+|C_p'|]$$

$$=|K_1|-[|K_1\setminus A|+|p|K_1|-\left\{\sum_{k=1}^{p}|T_{\omega_{N_k}}(K_2\cap A)|\right\}]$$

$$>|K_1|-[|K_1\setminus A|+p|K_1|-p\alpha|K_2\cap A|+p\mu]$$

$$>|K_1|-[|K_1\setminus A|+p|K_1|-p(1-\frac{1}{2p^2})(|K_2|-|K_2\setminus A|)+p\mu]$$

$$=|K_1|-|K_1\setminus A|+p|K_1|-p|K_2|+(p-\frac{1}{2p})|K_2\setminus A|+\frac{1}{2p}|K_2|+p\mu]$$

$$>|K_1|-[\varepsilon|K_1|+|K_2|+(p-\frac{1}{2p}\varepsilon|K_2|+\frac{1}{2p}|K_2|+\frac{1}{4p}|K_2|]$$

$$=|K_2|[\frac{p+1}{p}-\frac{4p+3}{4p}-\varepsilon\frac{2p^2+2p+1}{2p}]$$

$$=|K_2|[\frac{1}{4p}-\varepsilon.\frac{2p^2+2p+1}{2p}]$$

$$>0,\text{ since } 0<\varepsilon<\frac{1}{2(2p^2+2p+1)}.$$

Thus for $N_1, N_2, \dots, N_p \geqq N_0$, we obtain by (II), that

$$A\cap T_{\omega_{N_1}}(A)\cap T_{\omega_{N_2}}(A)\cap\dots\cap T_{\omega_{N_p}}(A)$$

is a set of positive measure.

Corollary. *If* $p=1$ *and* $|\det J|=1$, *then follows Theorem* 1 *of* [3].

For the next theorem we require the following condition:

(i) For every pair of points a, b in R_N and $\omega_n\in\Omega$, $\omega_n\to\omega_0$ (in Ω) and for any two spheres $K_1=S[a, r_1]$, $K_2=S[b, r_2]$ $(r_2<r_1)$ we have

$$\lim_{n\to\infty}[\sup\{|(a-T_{\omega_n}(K_2)|\}]=r_2.$$

Theorem 2. *Let A and B be two sets of positive measure in R_N and a and b be points of density of A and B respectively. Also let T_{ω_n} $(\omega_n\in\Omega)$ be transformations satisfying (i), (II) and (III)′ with respect to (a, b, ω_0). Then there exists a positive integer N_0 such that for $n\geqq N_0$, $A\cap T_{\omega_n}(B)$ is a set of positive measure.*

Proof. Since A and B are of positive measure, there exist spheres $K_1=S[a, r_1]$ and $K_2=S[b, r_2]$ $(r_2<r_1)$ such that

$$|K_1\setminus A|<\varepsilon|K_1| \text{ and } |K_2\setminus B|<\varepsilon|K_2|$$

where $0<\varepsilon<\frac{p^N}{2p^{N+1}+2p^N-1}$, $p(>1)$ being a number such that $p|K_2|\geqq|K_1|$.

Now, if $\sup\{|a-T_{\omega_n}(K_2)|\}=d_n$, then by condition (i) there exists a positive integer N_1 such that for $n\geqq N_1$

$$|d_n-r_2|<r_1-r_2.$$

Thus for $n\geqq N_1$, $T_{\omega_n}(K_2)\subset K_1$, so that $T_{\omega_n}(K_2\cap B)\subset K_1$. Also, by the condition similar to (III)′ there exists a positive integer N_2 such that for $n\geqq N_2$,

$$||T_{\omega_n}(K_2\cap B)-\alpha|K_2\cap B||<|K_2|\frac{p^N-1}{2p^N}=\lambda, \text{ say}$$

where $1-\frac{1}{2p^N}<\alpha<1+\frac{1}{2p^N}$.

Next, if $C=K_1\cap A$ and $C_n=T_{\omega_n}(K_2\cap B)$, then $C\cap C_n=K_1\setminus[C'\cup C_n']$ (complements being taken with respect to K_1). If $N_0=\max(N_1, N_2)$, then for $n\geqq N_0$,

$$\begin{aligned}|C\cap C_n| &\geqq |K_1|-[|C'|+|C_n'|]\\ &=|K_1|-[|K_1\setminus A|+|K_1|-|T_{\omega_n}(K_2\cap B)|]\\ &=|T_{\omega_n}(K_2\cap B)|-|K_1\setminus A|\\ &>\alpha|K_2\cap B|-\lambda|K_2|-|K_1\setminus A|\\ &=\alpha[|K_2|-|K_2\setminus B|]-\lambda|K_2|-|K_1\setminus A|\\ &>(1-\frac{1}{2p^N})|K_2|-(1-\frac{1}{2p^N})\varepsilon|K_2|-p\varepsilon|K_2|-\lambda|K_2|\\ &=|K_2|[1-\frac{1}{2p^N}-\varepsilon+\frac{1}{2p^N}\varepsilon-p\varepsilon-\frac{p^N-1}{2p^N}]\\ &=|K_2|\frac{p^N-(2p^{N+1}+2p^N-1)\ \varepsilon}{2p^N}\\ &>0, \text{ since } 0<\varepsilon<\frac{p^N}{2p^{N+1}+2p^N-1}.\end{aligned}$$

Finally, by (II) we have, for $n\geqq N_0$ is a set of positive measure $A\cap T_{\omega_n}(B)$.

Theorem 3. *Let $A\subset R_N$ be a bounded set of positive measure. Let the transformations $T_\omega(\omega\in\Omega)$ satisfy the condition (I). Then there exist a sphere K_0 (with centre at the origin) and a positive integer N_0 such that for every $x\in K_0$, there are vectors $a_0(x), a_{N_1}(x),\ldots, a_{N_p}(x)$ in A such that for $N_1, N_2,\ldots, N_p\geqq N_0$,*

$$a_0(x)=a_k(x)-T_{\omega_n}(x)\qquad(k=1,2,\ldots,p).$$

Proof. Since A is a set of positive measure, there exists a sphere $K=S[x_0, r]$ in R_N such that $|K\setminus A|<\varepsilon|K|$, where $0<\varepsilon<\frac{1}{4^N(p+1)}$.

To prove the theorem we shall prove that there exists a positive integer N_0 such that for $N_1, N_2,\ldots, N_p\geqq N_0$

$$X(x)=(A-x_0)\cap[A-x_0-T_{\omega_{N1}}(x)]\cap\ldots\cap[A-x_0-T_{\omega_{N_p}}(x)]$$

is a set of positive measure for every $x\in K_0=S[0, r/4]$.

Now, in view of condition (I) we can easily verify that there exists a positive integer N_0 such that for $N_1, N_2,\ldots, N_p\geqq N_0$,

$$S[x_0+T_{\omega_{N_k}}(x),\ r/4]\subset K$$

for every $x\in K_0=s[0,r/4]$ and $k=1,2,\ldots,p$.

If possible, let $|X(y)|=0$ for some $y\in K_0$. Then for $N_1,N_2,\ldots,N_p\geqq N_0$,

$$|K_0|=|S[0,\ r/4]|=|K_0\setminus X(y)|$$
$$=|K_0\setminus[(A-x_0)\cap(A-x_0-T_{\omega_{N1}}(y))\cap\ldots$$
$$\ldots\cap(A-x_0-T_{\omega_{N_p}}(Y))\,]|$$
$$\leqq|K_0\setminus A-x_0|+\sum_{k=1}^{p}|K_0\setminus(A-x_0-T_{\omega_{N_k}}(y))|$$
$$=|S[x_0,\ r/4]\setminus A+\sum_{k=1}^{p}|Sx_0+T_{\omega_{N_k}}(y),\ r/4]\setminus A|$$
$$\leqq|K\setminus A+\sum_{k=1}^{p}|K\setminus A|$$
$$=(p+1)\,|K\setminus A|$$
$$<(p+1)\varepsilon\,|K|$$
$$<\frac{|K|}{4^N},\ \text{since } 0<\varepsilon<\frac{1}{4^N(p+1)}.$$

This is clearly a contradiction. So, for each $x\in K_0$ and $N_1,N_2,\ldots,N_p\geqq N_0$, there exist vectors $a_0(x),a_1(x),a_2(x),\ldots,a_p(x)$ in A such that

$$a_0(x)-x_0=a_1(x)-x_0-T_{\omega_{N_1}}(x)=\ldots=a_p(x)-x_0-T_{\omega_{N_p}}(x)$$

i. e.,

$$a_0(x)=a_k(x)-T_{\omega_{N_k}}(x)\qquad(k=1,2,\ldots,p)$$

For the next theorem we require the following condition:

(IV). If A and B are measurable sets in R_N and $\omega_n\to\omega_0$ (in Ω), then

$$\lim_{n\to\infty}|A\cap T_{\omega_n}(B)|=|A\cap T_{\omega_0}(B)|=|A\cap B|.$$

Theorem 4. *Let A and B be two sets of positive measure in R_N and T_{ω_n} be a sequence of transformations in R_n satisfying the conditions (II) and (IV). If there is a common point of density of A and B, then there exists a positive integer N_0 such that for $N_1,N_2,\ldots,N_p\geqq N_0$*

$$A\cap T_{\omega_{N_1}}(B)\cap T_{\omega_{N_2}}(B)\cap\ldots\cap T_{\omega_{N_p}}(B)$$

is a set of positive measure.

Proof. Let α be a point of density of A and B. Then there exists a sphere $K=S[\alpha,r]$ such that $|K\setminus A|<\varepsilon|K|$ and $|K\setminus B|<\varepsilon|K|$, where $0<\varepsilon<\frac{1}{2(p+1)^2}$.

Now by condition (IV) there exists a positive integer N_0 such that for $n \geqq N_0$,

$$||K \cap T_{\omega_n}(B)| - |K \cap B|| < \frac{|K|}{2(p+1)} = \lambda, \text{ say,}$$

i. e.

$$|K \cap B| - \lambda < |K \cap T_{\omega_n}(B)| < |K \cap B| + \lambda.$$

Let $N_1, N_2, \dots, N_p \geqq N_0$ and $C = K \cap A$, $C_k = K \cap T_{\omega_{N_k}}(B)$ $(k = 1, 2, \dots, p)$. Also lex $X = C \cap C_1 \cap \dots \cap C_p$. Then

$$|X| \geqq |K| - [|K \setminus A| + |K \setminus T_{\omega_{N_1}}(B)| + \dots, + |K \setminus T_{\omega_{N_p}}(B)|]$$

$$= |K| - [|K \setminus A| + p|K| - |K \cap T_{\omega_{N_1}}(B)| - \dots - |K \cap T_{\omega_{N_p}}(B)|]$$

$$> |K| - [|K \setminus A| + p|K| - p|K \cap B| + p\lambda]$$

$$= |K| - [|K \setminus A| + p|K \setminus B| + p\lambda]$$

$$> |K| - [\varepsilon|K| + p\varepsilon|K| + p\lambda]$$

$$> |K| - \frac{1}{2}|K| > 0,$$

since $0 < \varepsilon < \frac{1}{2(p+1)^2}$ and $\lambda = \frac{|K|}{2(p+1)}$.

Thus by (II), for $N_1, N_2, \dots, N_p \geqq N_0$,

$$A \; T_{\omega_{N_1}}(B) \cap T_{\omega_{N_2}}(B) \cap \dots \cap T_{\omega_{N_p}}(B)$$

is a set of positive measure.

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+ *Department of Mathematics*
University of Kalyani
Kalyani – 741235
West Bengal
INDIA

++ *Sutragarh Girls' High School*
Santipur
West Bengal
INDIA

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