Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol.7, 1993, Fasc. 1

Quantum Diffusions on the Finite-Difference Fock Space

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Presented by Bl. Sendov

We provide an existence and uniqueness theorem for a class of quantum diffusions acting on a subalgebra of the algebra of bounded linear operators on the Finite-Difference Fock space.

1. Introduction

The existence and uniqueness of quantum diffusions acting on the algebra of bounded operators on the Heisenberg Fock space was proved by Evans [6] with the use of the stochastic calculus of Hudson and Parthasarathy [8].

The Finite-Difference (FD) Fock space F, which is based on the FD Lie algebra of Feinsilver [7], was constructed in [2], [3] and it was equipped with a stochastic calculus in [4].

In brief, if S denotes the set of step functions $[0,\infty) \to (-1,1)$, the FD Fock space F is defined as the Hilbert space completion of $E \stackrel{\text{def}}{=} \text{span}\{y(f) | f \in S\}$ with $< y(f), y(g) > = \exp(-\int_0^\infty \log(1 - f(s)g(s))ds)$. The $y(f)$'s are called the "exponential vectors". For each $f \in S$ the FD operators $P(f), Q(f), T(f) : E \to F$ defined by

$$Q(f)y(g) = \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} (y(g+\epsilon f) + y(e^{\epsilon f}\, g)) \text{ (weak derivative)}$$

$$P(f)y(g) = (\int_0^\infty f(t)g(t)\,dt + Q(fg))\,y(g)$$

$$T(f)y(g) = (Q(f) + P(f) + \int_0^\infty f(t)dt)\,y(g)$$

satisfy

$$[P(f), Q(g)] = [P(f), T(g)] = [T(f), Q(g)] = T(fg)$$

which are the commutation relations of the FD algebra.

The basic integrator processes of the FD stochastic calculus are "time" t and

$$M_{i,j,k}(t) \stackrel{\text{def}}{=} Q^{(i)}(\chi_{[0,t]})T^{j}(\chi_{[0,t]})P^{(k)}(\chi_{[0,t]})$$

where $t \geq 0$, $(i,j,k) \in \mathbb{N}_0^3$ and $x^{(i)} = x(x-1)\ldots(x-i+1)$, $x^{(0)} = 1$. Their stochastic differentials can be multiplied using Ito's formula: $dM_{i,j,k}(t) \cdot dt = dt \cdot dM_{i,j,k}(t) = 0$ and for $(i,j,k),(I,J,K) \in \mathbb{N}_0^3$

$$\begin{aligned} dM_{i,j,k}(t) \cdot dM_{I,J,K}(t) = & \sum_{\lambda=0}^{I} \sum_{\mu=0}^{k} \sum_{\nu=0}^{\lambda} \sum_{\tau=0}^{\mu} \sum_{\sigma=0}^{\min(i,\nu)} \sum_{\rho=0}^{\min(K,\tau)} \delta_{k-\mu,I-\lambda} \\ & \times \binom{I}{\lambda}\binom{k}{\mu}\binom{\lambda}{\nu}\binom{\mu}{\tau}\binom{i}{\sigma}\binom{\nu}{\sigma}\binom{K}{\rho}\binom{\tau}{\rho} \\ & \times (k-\mu)! j^{(\lambda-\nu)} J^{(\mu-\tau)} \sigma!\, \rho!\, dM_{i+\ni-\sigma, j+J+I-\lambda, K+\tau-\rho}(t) \\ & - M_{i,j,k}(t)\, dM_{I,J,K}(t) - M_{I,J,K}(t)\, dM_{i,j,k}(t) \end{aligned}$$

If $f = \sum_{\lambda=1}^{n} a_\lambda \chi_{I_\lambda}$, $g = \sum_{\lambda=1}^{n} b_\lambda \chi_{I_\lambda} \in S$, $(i,j,k) \in \mathbb{N}_0^3$, δ is Kronecker's delta, $\alpha, \beta, \gamma \in \{0,1\}$,

$$\sigma_{i,j,k}^{f,g}(s) \stackrel{\text{def}}{=} \begin{cases} \delta_{i,0}\, \delta_{k,0}\, (\delta_{j,0} + (1-\delta_{j,0})(j-1)!), \text{ if } s \notin \bigcup_{\lambda=1}^{n} I_\lambda \\ (1-\delta_{i+j+k,0}) \cdot (i+j+k-1)! \frac{f^k(s) g^i(s) (1+f(s))^{i+j} (1+g(s))^{j+k}}{(1-f(s)g(s))^{i+j+k}}, \\ \text{if } s \in \bigcup_{\lambda=1}^{n} I_\lambda \end{cases}$$

$$\Phi_{i,j,k}^{\alpha,\beta,\gamma} \stackrel{\text{def}}{=} (1-\delta_{\alpha+i,0})(1-\delta_{\beta+j,0})(1-\delta_{\gamma+k,0}) \cdot$$

$$\cdot [(1-\delta_{i\delta_{\alpha,0},0})(1-\delta_{j\delta_{\beta,0},0})(1-\delta_{k\delta_{\gamma,0},0}) + (1-\delta_{i\delta_{\alpha,0},0})(1-\delta_{j\delta_{\beta,0},0})\delta_{k\delta_{\gamma,0},0} +$$

$$+ (1-\delta_{i\delta_{\alpha,0},0})\delta_{j\delta_{\beta,0},0}(1-\delta_{k\delta_{\gamma,0},0}) + \delta_{i\delta_{\alpha,0},0}(1-\delta_{j\delta_{\beta,0},0})(1-\delta_{k\delta_{\gamma,0},0}) +$$

$$+ (1-\delta_{i\delta_{\alpha,0},0})\delta_{j\delta_{\beta,0}+k\delta_{\gamma,0},0} + (1-\delta_{j\delta_{\beta,0},0})\delta_{i\delta_{\alpha,0}+k\delta_{\gamma,0},0} + (1-\delta_{k\delta_{\gamma,0},0})\delta_{i\delta_{\alpha,0}+j\delta_{\beta,0},0}]$$

and $X = \{X(s) : E \rightarrow F \,|\, s \geq 0\}$ is an adapted process, in the sense of Hudson and Parthasarathy [8], for which

$$\sum_{\alpha,\beta,\gamma \in \{0,1\}} \Phi_{i,j,k}^{\alpha,\beta,\gamma} \sum_{I=1}^{i\delta_{\alpha,0}} \sum_{J=1}^{j\delta_{\beta,0}} \sum_{K=1}^{k\delta_{\gamma,0}} \binom{i\delta_{\alpha,0}}{I}\binom{j\delta_{\beta,0}}{J}\binom{k\delta_{\gamma,0}}{K} \cdot \int_0^t \sigma_{I,J,K}^{f,g}(s)$$

$$\times < X(s)\, M_{\alpha i, \beta j, \gamma k}(s)\, M_{i\delta_{\alpha,0}-I, j\delta_{\beta,0}-J, k\delta_{\gamma,0}-K}(s)\, y(f), y(g) > \; ds$$

makes sense, then X is called $M_{i,j,k}$-integrable and the expression above is denoted by $< \int_0^t X(s)\, dM_{i,j,k}(s) y(f), y(g) >$.

If A is a unital $\star$-subalgebra of $B(F)$ (the algebra of bounded linear operators: $F \to F$) then a "quantum diffusion on A" is a family $\{\delta_t(x) : A \to B(F) \,|\, t \geq 0\}$ of identity preserving contractive $\star$-homomorphisms satisfying for each $x \in A$:

$$\delta_t(x)\,h \;=\; xh \;+\; \sum_{(i,j,k)\in\Lambda} \int_0^t \delta_s(\lambda_{i,j,k}(x))\,dM_{i,j,k}(s)\,h \text{ for all } h \in E.$$

Here Λ is a finite subset of $\mathbb{N}_0^3$ and the $\lambda_{i,j,k}$'s are maps from A into itself called the "structure maps". In order for such a family to exist the structure maps must satisfy certain conditions known as the "structure equations". The method used in [6] to derive the structure equations in the Heisenberg case was based on the fact that the basic integrator processes were linearly independent in the sense of Accardi, Fagnola and Quaegebeur [1] along with an injectivity assumption on the δ_t's. Due to the complexity of the Ito's formula in the FD case, Evans' method is difficult to employ and so we replace the structure equations with an iterated integral condition.

We remark finally that $M_{0,1,0}$ is a quantum exponential process (cf: [2], [3], [4]) and the study of such diffusions appears naturally in the study of quantum systems in the presence of "exponential noise".

2. A fundamental integration formula

The most commonly used method for obtaining strong solutions of quantum SDE's is Picard's method of successive approximations which in order to be employed requires, in our case, the existence of a bound on expressions like $\| \int_0^t X(s)\,dM_{i,j,k}(s)\,y(f)\|$ where $t \geq 0$, $(i,j,k) \in \mathbb{N}_0^3$, $f \in S$ and X is a $M_{i,j,k}$-integrable adapted process.

To get such a bound we provide a formula for the combined matrix element $< \int_0^t X(s)\,dM_{i,j,k}(s)\,y(f)\,,\, \int_0^t Y(s)\,dM_{I,J,K}(s)\,y(g) >$ where $(I,J,K) \in \mathbb{N}_0^3$, $g \in S$ and Y is a $M_{I,J,K}$-integrable adapted process. The proof of the formula relies on the following lemma whose proof is based on the "independent increments" property of the FD algebra (cor. 1.13.1 of [2]) and is similar to that of lemma 3.2 of [4].

Lemma 2.1. *Let* $f = \sum_{\lambda=1}^n \alpha_\lambda \chi_{A_\lambda}$, $g = \sum_{\lambda=1}^n \beta_\lambda \chi_{A_\lambda} \in S$, *let* A, B *be disjoint subsets of* $[0,\infty)$ *with finite Lebesgue measures* $\mu(A)$, $\mu(B)$ *and let* (i,j,k), $(I,J,K) \in \mathbb{N}_0^3$. *Then:*

$$
< Q^{(i)}(x_A)\,T^j(x_A)\,P^{(k)}(x_A)\,Q^{(I)}(x_B)\,T^J(x_B)\,P^{(K)}(x_B)\,y(f)\,,\;y(g) >
$$

$$
= \begin{cases}
\delta_{i,0}\,\delta_{k,0}\,\delta_{I,0}\,\delta_{K,0}(\mu(A))_j(\mu(B))_J < y(f), y(g) > , \text{ if } (A\cup B)\cap(\cup_{\lambda=1}^n A_\lambda)=\emptyset \\
\delta_{i,0}\delta_{k,0}(\mu(A))_j(\mu(B))_{I+J+K}\dfrac{\alpha_{\lambda_0}^K\beta_{\lambda_0}^I(1+\alpha_{\lambda_0})^{I+J}(1+\beta_{\lambda_0})^{J+K}}{(1-\alpha_{\lambda_0}\beta_{\lambda_0})^{I+J+K}} < y(f), y(g) > \\
\qquad \text{if } B\subset A_{\lambda_0} \text{ for some } \lambda_0\in\{1,2,\dots,n\} \text{ and } A\cap(\cup_{\lambda=1}^n A_\lambda)=\emptyset \\
\delta_{I,0}\,\delta_{K,0}(\mu(B))_J(\mu(A))_{i+j+k}\,\dfrac{\alpha_{\lambda_0}^k\beta_{\lambda_0}^i(1+\alpha_{\lambda_0})^{i+j}(1+\beta_{\lambda_0})^{j+k}}{(1-\alpha_{\lambda_0}\beta_{\lambda_0})^{i+j+k}} < y(f), y(g) > \\
\qquad \text{if } A\subset A_{\lambda_0} \text{ for some } \lambda_0\in\{1,2,\dots,n\} \text{ and } B\cap(\cup_{\lambda=1}^n A_\lambda)=\emptyset \\
(\mu(A))_{i+j+k}(\mu(B))_{I+J+K}\,\dfrac{\alpha_{\lambda_0}^k\beta_{\lambda_0}^i(1+\alpha_{\lambda_0})^{i+j}(1+\beta_{\lambda_0})^{j+k}}{(1-\alpha_{\lambda_0}\beta_{\lambda_0})^{i+j+k}} \\
\cdot\qquad \dfrac{\alpha_{\ell_0}^K\beta_{\ell_0}^I(1+\alpha_{\ell_0})^{I+J}(1+\beta_{\ell_0})^{J+K}}{(1-\alpha_{\ell_0}\beta_{\ell_0})^{I+J+K}} < y(f), y(g) > \\
\qquad \text{if } A\subset A_{\lambda_0} \text{ and } B\subset A_{\lambda_0} \text{ for some } \lambda_0,\ell_0\in\{1,2,\dots,n\},\ \lambda_0\neq\ell_0.
\end{cases}
$$

Here $(\mu)_n=\mu(\mu+1)\dots(\mu+n-1)$ *and* $(\mu)_0=1$.

Theorem 2.1. *If* $(i,j,k),(I,J,K)\in\mathbb{N}_0^3$ *and* X,Y *are respectively* $M_{i,j,k}$, $M_{I,J,K}$*-integrable adapted processes then for all* $f,g\in S$ *and* $t\geq 0$*:*

$$
< \int_0^t X(s)\,dM_{i,j,k}(s)\,y(f)\,,\;\int_0^t Y(s)\,dM_{I,J,K}(s)\,y(g) > =
$$

$$
= \sum_{\alpha,\beta,\gamma\in\{0,1\}}\ \sum_{a,b,c\in\{0,1\}} \Phi_{i,j,k}^{\alpha,\beta,\gamma}\Phi_{I,J,K}^{a,b,c} \sum_{i'=1}^{i\delta_{\alpha,0}}\sum_{j'=1}^{j\delta_{\beta,0}}\sum_{k'=1}^{k\delta_{\gamma,0}}\sum_{I'=1}^{I\delta_{a,0}}\sum_{J'=1}^{J\delta_{b,0}}\sum_{K'=1}^{K\delta_{c,0}}\binom{i\delta_{\alpha,0}}{i'}
$$

$$
\times\binom{j\delta_{\beta,0}}{j'}\binom{k\delta_{\gamma,0}}{k'}\binom{I\delta_{a,0}}{I'}\binom{J\delta_{b,0}}{J'}\binom{K\delta_{c,0}}{K'}\left[\sum_{\lambda=0}^{i'}\sum_{\mu=0}^{I'}\sum_{\nu=0}^{\lambda}\sum_{\tau=0}^{\mu}\sum_{\sigma=0}^{\min(K',\nu)}\right.
$$

$$
\times\sum_{\rho=0}^{\min(K',\tau)}\delta_{I'-\mu,i'-\lambda}\binom{i'}{\lambda}\binom{I'}{\mu}\binom{\lambda}{\nu}\binom{\mu}{\tau}\binom{K'}{\sigma}\binom{\nu}{\sigma}\binom{k'}{\rho}\binom{\tau}{\rho}(I'-\mu)!
$$

$$
\times J'^{(\lambda-\nu)}j'^{(\mu-\tau)}\sigma!\,\rho!\int_0^t \sigma^{f,g}_{K'+\ni-\sigma,J'+j'+i'-\lambda,k'+\tau-\rho}(s) < X(s)M_{\alpha i,\beta j,\gamma k}(s)
$$

$$
\times M_{i\delta_{\alpha,0}-i',j\delta_{\beta,0}-j',k\delta_{\gamma,0}-k'}(s)y(f)\,,\,Y(s)M_{aI,bJ,cK}(s)M_{I\delta_{a,0}-I',J\delta_{b,0}-J',K\delta_{c,0}-K'}(s)y(g) > ds+
$$

$$
+\int_0^t\int_0^t \sigma^{f,g}_{i',j',k'}(s)\,\sigma^{f,g}_{K',J',I'}(\omega) < X(s)M_{\alpha i,\beta j,\gamma k}(s)M_{i\delta_{\alpha,0}-i',j\delta_{\beta,0}-j',k\delta_{\gamma,0}-k'}(s)y(f)\,,
$$

$$
\left. Y(\omega)M_{aI,bJ,cK}(\omega)M_{I\delta_{a,0}-I',J\delta_{b,0}-J',K\delta_{c,0}-K'}(\omega)y(g) > ds\,d\omega\right].
$$

Proof. As in lemma 4.1 of [4] let

$dm_{i,j,k}(s)=(d(Q^{(i)})(s)+\delta_{i,0}\cdot Id)\cdot(d(T^j)(s)+\delta_{j,0}\cdot Id)\cdot(d(P^{(k)})(s)+\delta_{k,0}\cdot Id)$

where $Id: E\to\dot{E}$ is the identity operator. Then by definition 4.1 of [4]

$$< \int_0^t X(s)\,dM_{i,j,k}(s)\,y(f)\ ,\ \int_0^t Y(s)\,dM_{I,J,K}(s)\,y(g) > =$$

$$= \sum_{\alpha,\beta,\gamma\in\{0,1\}}\ \sum_{a,b,c\in\{0,1\}} (1-\delta_{\alpha+i,0})(1-\delta_{\beta+j,0})(1-\delta_{\gamma+k,0})(1-\delta_{a+I,0})$$

$$\times (1-\delta_{b+J,0})(1-\delta_{c+K,0}) \cdot\ < \int_0^t X(s)\, M_{\alpha i,\beta j,\gamma k}(s)\, dm_{i\delta_{\alpha,0},j\delta_{\beta,0},k\delta_{\gamma,0}}(s)\, y(f)\ ,$$

$$\int_0^t Y(s)\, M_{aI,bJ,cK}(s)\, dm_{I\delta_{a,0},J\delta_{b,0},K\delta_{c,0}}(s)\, y(g) > =$$

$$= \sum_{\alpha,\beta,\gamma\in\{0,1\}}\ \sum_{a,b,c\in\{0,1\}} (1-\delta_{\alpha+i,0})(1-\delta_{\beta+j,0})(1-\delta_{\gamma+k,0})(1-\delta_{a+I,0})$$

$$\times (1-\delta_{b+J,0})(1-\delta_{c+K,0}) \cdot \lim_{|\pi|\to 0} \sum_{\ell=0}^{n-1}\sum_{\lambda=0}^{n-1} < X(s_\ell\, M_{\alpha i,\beta j,\gamma k}(s_\ell)$$

$$\times \Big[d(Q^{(i\delta_{\alpha,0})})(s_\ell)d(T^{j\delta_{\beta,0}})(s_\ell)d(P^{(k\delta_{\gamma,0})})(s_\ell) + \delta_{k\delta_{\gamma,0},0}\, d(Q^{(i\delta_{\alpha,0})})(s_\ell)d(T^{j\delta_{\beta,0}})(s_\ell)+$$

$$+\ \delta_{j\delta_{\beta,0},0}\, d(Q^{(i\delta_{\alpha,0})})(s_\ell)d(P^{(k\delta_{\gamma,0})})(s_\ell) + \delta_{j\delta_{\beta,0},0+k\delta_{\gamma,0},0}\, d(Q^{(i\delta_{\alpha,0})})(s_\ell)+$$

$$+\ \delta_{i\delta_{\alpha,0},0}\, d(T^{j\delta_{\beta,0}})(s_\ell)d(P^{(k\delta_{\gamma,0})})(s_\ell) + \delta_{i\delta_{\alpha,0},0+k\delta_{\gamma,0},0}\, d(T^{j\delta_{\beta,0}})(s_\ell)+$$

$$+\ \delta_{i\delta_{\alpha,0},0+j\delta_{\beta,0},0}\, d(P^{(k\delta_{\gamma,0})})(s_\ell)\Big]\ y(f)\ ,$$

$$Y(s)_\lambda\, M_{aI,bJ,cK}(s_\lambda)\ \Big[d(Q^{(I\delta_{a,0})})(s_\lambda)d(T^{J\delta_{b,0}})(s_\lambda)d(P^{(K\delta_{c,0})})(s_\lambda)+$$

$$+\ \delta_{K\delta_{c,0},0}\, d(Q^{(I\delta_{a,0})})(s_\lambda)d(T^{J\delta_{b,0}})(s_\lambda) + \delta_{J\delta_{b,0},0}\, d(Q^{(I\delta_{a,0})})(s_\lambda)d(P^{(K\delta_{c,0})})(s_\lambda)+$$

$$+\ \delta_{J\delta_{b,0},0+K\delta_{c,0},0}\, d(Q^{(I\delta_{a,0})})(s_\lambda) + \delta_{I\delta_{a,0},0}\, d(T^{J\delta_{b,0}})(s_\lambda)d(P^{(K\delta_{c,0})})(s_\lambda)+$$

$$+\ \delta_{I\delta_{a,0},0+K\delta_{c,0},0}\, d(T^{J\delta_{b,0}})(s_\lambda) + \delta_{I\delta_{a,0},0+J\delta_{b,0},0}\, d(P^{(K\delta_{c,0})})(s_\lambda)\Big]\ y(g) >$$

where $\pi = \{0 = s_0, s_1, s_2, \ldots, s_n = t\}$ is a partition of $[0,t]$ and the proof is completed in a way similar to that of the proof of lemma 4.1 of [4] by using the linearity of $<,>$, distinguishing cases $\lambda = \ell$ and $\lambda \neq \ell$ and applying lemma 2.1 to compute each term.

3. The structure maps

Let $\Lambda \subset \mathbb{N}_0^3$ be a set of cardinality $|\Lambda| < \infty$ such that $(i,j,k) \in \Lambda \Leftrightarrow (k,j,i) \in \Lambda$ and let $A \subset B(F)$ be a unitial $\star$-algebra. We assume the existence of a family $\{\lambda_{i,j,k} : A \to A/(i,j,k) \in \Lambda\}$ of "structure maps" such that:

(a) $\lambda_{i,j,k}$ is linear.
(b) For each $x \in A, [\lambda_{i,j,k}(x)]^\star = \lambda_{k,j,i}(x^\star)$.
(c) $\lambda_{i,j,k}(I) = 0$, where $I \in A$ is the identity.
(d) For each $x \in A, f \in S$, $\alpha,\beta,\gamma \in \{0,1\}$, $1 \leq i' \leq i\delta_{\alpha,0}$, $1 \leq j' \leq j\delta_{\beta,0}$, $1 \leq k' \leq k\delta_{\gamma,0}$ and $t \geq 0$ there exists $v_t \geq 0$ (depending on them) such

that for every $B = \{B_t : A \to L(E,F)/t \geq 0\}$ with $\{B_t(x)/t \geq 0\}$, a strongly continuous adapted process

$$\|B_t(\lambda_{i,j,k}(x))M_{\alpha i,\beta j,\gamma k}(t)M_{i\delta_{\alpha,0}-i',j\delta_{\beta,0}-j',k\delta_{\gamma,0}-k'}(t)y(f)\| \leq v_t\|B_t(x)y(f)\|.$$

Here $L(E,F)$ is the space of linear operators $E \to F$.

(e) For each $\tau \geq 0$, $v_\tau \stackrel{\text{def}}{=} \sup_{0\leq t\leq \tau} v_t < \infty$
For each $\tau \geq 0$ and $f \in S$, we define

$$\xi \stackrel{\text{def}}{=} 2|\Lambda| \sum_{(i,j,k)\in\Lambda} \sum_{\alpha,\beta,\gamma\in\{0,1\}} \Phi_{i,j,k}^{\alpha,\beta,\gamma}\cdot$$
$$\times \sum_{i'=1}^{i\delta_{\alpha,0}} \sum_{j'=1}^{j\delta_{\beta,0}} \sum_{k'=1}^{k\delta_{\gamma,0}} \sum_{I'=1}^{i\delta_{\alpha,0}} \sum_{J'=1}^{j\delta_{\beta,0}} \sum_{K'=1}^{k\delta_{\gamma,0}} \binom{i\delta_{\alpha,0}}{i'}\binom{j\delta_{\beta,0}}{j'}\binom{k\delta_{\gamma,0}}{k'}\binom{i\delta_{\alpha,0}}{I'}$$
$$\times \binom{j\delta_{\beta,0}}{J'}\binom{k\delta_{\gamma,0}}{K'}\Bigg[\sum_{\lambda=0}^{i'}\sum_{\mu=0}^{I'}\sum_{\nu=0}^{\lambda}\sum_{\tau=0}^{\mu}\sum_{\sigma=0}^{\min(K',\nu)}\sum_{\rho=0}^{\min(K',\tau)} \delta_{I'-\mu,i'-\lambda}\binom{i'}{\lambda}\binom{I'}{\mu}$$
$$\times \binom{\lambda}{\nu}\binom{\mu}{\tau}\binom{K'}{\sigma}\binom{\nu}{\sigma}\binom{k'}{\rho}\binom{\tau}{\rho}(I'-\mu)!J'^{(\lambda-\nu)}j'^{(\mu-\tau)}\sigma!\rho!$$
$$\times \sup_{0\leq s\leq\tau}|\sigma^{f,f}_{K'+\nu-\sigma,J'+j'+i'-\lambda,k'+\tau-\rho}(s)| + \tau \sup_{0\leq s,\omega\leq\tau}|\sigma^{f,f}_{i',j',k'}(s)\sigma^{f,f}_{K',J',I'}(\omega)|\Bigg]\nu_\tau^2$$

(f) With $a_n \stackrel{\text{def}}{=} (i_n, j_n, k_n) \in \Lambda$, we assume that for all $x, y \in A$, $t \geq 0$ and $f, g \in S$:

$$< \left[xy + \sum_{k=1}^{\infty}\sum_{a_1,\ldots,a_k\in\Lambda}\int_0^{t_0}\cdots\int_0^{t_{k-1}} \lambda_{a_k}\ldots\lambda_{a_1}(xy)dM_{a_k}(t_k)\ldots dM_{a_1}(t_1)\right] y(f), y(g) >$$
$$- < \left[y + \sum_{\ell=1}^{\infty}\sum_{a'_1,\ldots,a'_\ell\in\Lambda}\int_0^{t_0}\cdots\int_0^{t_{\ell-1}} \lambda_{a'_\ell}\ldots\lambda_{a'_1}(y)dM_{a'_\ell}(t_\ell)\ldots dM_{a'_1}(t_1)\right] y(f),$$
$$\left[x^* + \sum_{m=1}^{\infty}\sum_{a''_1,\ldots,a''_m\in\Lambda}\int_0^{t_0}\cdots\int_0^{t_{m-1}} \lambda_{a''_m}\ldots\lambda_{a''_1}(x^*)dM_{a''_m}(t_m)\ldots dM_{a''_1}(t_1)\right] y(g) > = 0$$

where $t_0 = t$ and the infinite sums converge in the weak sense.

4. The existence and uniqueness theorem

Lemma 4.1. (Gronwall's inequality): *Let $\lambda \geq 0$ and let $f, g : [a, b] \to [0, \infty)$ be continuous and such that $f(t) \leq \lambda + \int_a^t f(s)g(s)ds$ for $a \leq t \leq b$. Then $f(t) \leq \lambda \cdot \exp(\int_a^t g(s)ds)$ for $a \leq t \leq b$.*

Proof. The proof can be found in [5].

Lemma 4.2 *Let $\tau \geq 0$, and for each $(i, j, k) \in \Lambda$ and $x \in A$ let $\{A_s(x)|0 \leq s \leq \tau\}$ be a strongly continuous adapted process (i. e. for each $f \in S$ the map $s \to A_s(x)y(f)$ is continuous). Then for each $f \in S$ and $t \in [o, \tau]$:*

$$\| \int_0^t A_s(\lambda_{i,j,k}(x))dM_{i,j,k}(s)y(f)\| \leq (\tau\zeta_{i,j,k})^{1/2} \sup_{0\leq t\leq\tau} \|A_t(x)y(f)\|$$

where $\zeta_{i,j,k}$ is defined by $\xi = 2|\Lambda| \sum_{(i,j,k)\in\Lambda} \zeta_{i,j,k}$.

Proof. By theorem 2.1

$$\begin{aligned}
&\| \int_0^t A_s(\lambda_{i,j,k}(x))\, dM_{i,j,k}(s)\, y(f)\|^2 \\
&=< \int_0^t A_s(\lambda_{i,j,k}(x))\, dM_{i,j,k}(s)\, y(f), \int_0^t A_s(\lambda_{i,j,k}(x))\, dM_{i,j,k}(s)\, y(f) > \\
&\leq \zeta_{i,j,k} \int_0^t \|A_s(x)y(f)\|^2 ds \leq \tau\zeta_{i,j,k}(\sup_{0\leq t\leq\tau} \|A_t(x)y(f)\|)^2.
\end{aligned}$$

Theorem 4.1 *Let A, Λ and $\{\lambda : A \to A \,/\, (i, j, k) \in \Lambda\}$ be as in section 3. There exists a unique family $\{\delta_t : A \to L(E, F)/t \geq 0\}$ such that*

(a) For each $x \in A$, $\{\delta_t(x)/t \geq 0\}$ is a strongly continuous adapted process, linear in x.
(b) For each $x \in A$, $\delta_0(x) = x$ and for $t \geq 0$ $[\delta_t(x)]^* = \delta_t(x^*)$.
(c) For each $t \geq 0$ $\delta_t(I)$, where I is the identity of A.
(d) For each $t \geq 0$ and $x \in A$,
$\delta_t(x) = x + \sum_{(i,j,k)\in\Lambda} \int_0^t \delta_s(\lambda_{i,j,k}(x))dM_{i,j,k}(s)$ in the strong (pointwise) sense.
(e) For each $x \in A$ and $t \geq 0$, $\delta_t(x)$ extends to an operator in $B(F)$ such that $\|\delta_t(x)\| \leq \|x\|$.
(f) For all $x, y \in A$ and $t \geq 0$ $\delta_t(xy) = \delta_t(x)\delta_t(y)$.

Proof. Let $\tau \geq 0$. For $t \in [0, \tau]$ and $x \in A$ define $\{\delta^\tau_{t,n}(x)\}_{n=0}^\infty$ by the strong equalities $\delta^\tau_{t,0}(x) = x$ and for $n \geq 1$

$$\delta^\tau_{t,n}(x) = x + \sum_{(i,j,k)\in\Lambda} \int_0^t \delta_{s,n-1}(\lambda_{i,j,k}(x))\, dM_{i,j,k}(s).$$

Clearly $\{\delta^{\tau}_{t,0}(x)|t \geq 0\}$ is an adapted strongly uniformly continuous process on $[0,\tau]$. By theorem 2.1

$$\|(\delta^{\tau}_{t,n}(x) - \delta_{t',n}(x))y(f)\|^2 \leq \xi \sup_{0\leq s\leq \tau} \|\delta^{\tau}_{s,n-1}(x)y(f)\|^2 \, |t-t'|$$

and so, by induction, $\{\delta^{\tau}_{t,n}(x)|t \geq 0\}$ is an adapted strongly uniformly continuous process on $[0,\tau]$ for all n. Moreover, using theorem 2.1, we obtain iteratively

$$\sup_{0\leq t\leq \tau} \|(\delta^{\tau}_{t,n}(x) - \delta^{\tau}_{t,n-1}(x)y(f)\| \leq \frac{(\sqrt{\xi\tau})^n}{\sqrt{n!}}\|x\| \cdot \|y(f)\|$$

Thus $\sum_{n=1}^{\infty} \|(\delta^{\tau}_{t,n}(x) - \delta^{\tau}_{t,n-1}(x))y(f)\|$ converges for each $x \in A$ and $f \in S$ uniformly on $[0,\tau]$ and we may define $\{\delta^{\tau}_t(x)|0 \leq t \leq \tau\}$ as the uniform limit $\delta^{\tau}_t(x)y(f) \stackrel{\text{def}}{=} \lim_n \delta^{\tau}_{t,n}(x)y(f)$. The process $\{\delta^{\tau}_t(x)|0 \leq t \leq \tau\}$ is adapted and strongly continuous. By the triangle inequality

$$\begin{aligned}
&\|(\delta^{\tau}_t(x) - x - \sum_{(i,j,k)\in\Lambda} \int_0^t \delta^{\tau}_s(\lambda_{i,j,k}(x))\, dM_{i,j,k}(s))y(f)\| \\
&\leq \|(\delta^{\tau}_t(x) - \delta^{\tau}_{t,n+1}(x))y(f)\| + \sum_{(i,j,k)\in\Lambda} \| \int_0^t (\delta^{\tau}_{s,n} - \delta^{\tau}_s)(\lambda_{i,j,k}(x))\, dM_{i,j,k}(s))y(f)\| \\
&\leq \sup_{0\leq t\leq\tau} \|(\delta^{\tau}_t(x) - \delta^{\tau}_{t,n+1}(x))y(f)\| + \tau^{1/2} \sum_{(i,j,k)\in\Lambda} \zeta^{1/2}_{i,j,k} \sup_{0\leq t\leq\tau} \|(\delta^{\tau}_{t,n} - \delta^{\tau}_t)(x)y(f)\|
\end{aligned}$$

(by lemma 4.2) which goes to zero as $n \to \infty$.

Thus $\delta^{\tau}_t(x) = x + \sum_{(i,j,k)\in\Lambda} \int_0^t \delta^{\tau}_s(\lambda_{i,j,k}(x))\, dM_{i,j,k}(s)$ on $[0,\tau]$ and if $\tilde{\delta}^{\tau}_t(x)$ has the same property then

$$\|(\delta^{\tau}_t(x) - \tilde{\delta}^{\tau}_t(x))y(f)\|^2 \leq \xi \int_0^t \|(\delta^{\tau}_s(x) - \tilde{\delta}^{\tau}_s(x))y(f)\|^2\, ds$$

and by lemma 4.1 $\delta^{\tau}_t(x) = \tilde{\delta}^{\tau}_t(x)$ on E. For each $x \in A$ and $t \geq 0$ let $\delta_t(x) = \delta^{\tau}_t(x)$ where $\tau \geq 0$ is such that $0 \leq t \leq \tau$.

The family $\{\delta_t|t \geq 0\}$ satisfies (a)–(d). By using condition (f) of section 3 and the definition of $\delta^{\tau}_t(x)$ we see that for each $x, y \in A$, $f, g \in S$ and $t \geq 0$

$$< \delta_t(xy)y(f)\, ,\, y(g) > \; = \; < \delta_t(y)y(f)\, ,\, \delta_t(x^*)y(g) > .$$

Moreover, by theorem 2.1 and lemma 4.1

$$\begin{aligned}
\|\delta_t(x)y(f)\|^2 &\leq 2\|x\|^2\|y(f)\|^2 + \xi \int_0^t \|\delta_s(x)y(f)\|^2 ds \\
&\leq 2\|x\|^2\|y(f)\|^2 \exp(\xi \int_0^t ds)
\end{aligned}$$

and so for every $x \in A$ and $f \in S$

$$\sup_{0 \leq t \leq \tau} \|\delta_t(x) y(f)\| \leq \sqrt{2} \|x\| \, \|y(f)\| \exp(\xi \tau / 2)$$

Using this bound Evans' proof of proposition 5.1 of [6] carries word for word to the FD case to prove (e) and (f).

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Received 29. VII. 1991