Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol.7, 1993, Fasc. 1

On Certain Transformations of Generalized Fractional q-integrals, II

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Recently, in [5] we defined and studied two generalized fractional q-integral operators which generalize the known fractional q-integral operators due to W. A. Al-Salam [2], R. P. Agarwal [1], M. Upadhyay [7], W. A. Al-Salam and A. Verma [3] and the author [4]. Further, in [6] we obtained certain transformations involving these generalized fractional q-integral operators. The aim of the present paper is to obtain some more transformations involving these operators.

1. Introduction

Recently, the author [5] defined and studied the following two generalized fractional q-integrals:

$$I_q[(a);(b);\omega,\lambda,z,\mu;\eta:f(x)] = \frac{x^{-\eta\lambda-\lambda}}{(1-q)}\int_0^x t^{\eta\lambda+\lambda-1}{}_A\Phi_B^{(q^\lambda)}\left[\begin{matrix}(a); & \omega^\lambda z^\mu t^\mu/x^\mu\\ (b); & \end{matrix}\right] f(t)d(t;q) \tag{1}$$

and

$$K_q[(a);(b);\omega,\lambda;z,\mu;\eta:f(x)] = \frac{(x/q)^{\eta\lambda+\lambda-1}}{(1-q)}\int_x^\infty t^{-\eta\lambda-\lambda}{}_A\Phi_B^{(q^\lambda)}\left[\begin{matrix}(a); & \omega^\lambda z^\mu x^\mu/t^\mu\\ (b); & \end{matrix}\right] f(t)d(t;q). \tag{2}$$

(i) For $\lambda=\mu=\omega=1$, the operators (1) and (2) reduce to the operators due to M. Upadhyay [7].

(ii) For $\lambda=1$, $\mu=m$, $\omega=q^{\alpha-1}$, $B=0$, $A=1$, $a_1=-\alpha+1$ and $z=q$, the operator (1) reduces to an operator due to the author [4] which on further putting $m=1$ becomes an operator due to R. P. Agarwal [1].

(iii) For $B=0$, $A=1$, $a_1=-\alpha+1$, $\lambda=1$, $\mu=m$, $\omega=q^{\alpha-1}$, $z=1$ and $f(x)$ replaced by $f(xq^{1-\alpha})$, (2) reduces to an operator due to the author [4] which on further putting $m = 1$ becomes an operator due to W. A. Al-Salam [2].

(iv) For $\lambda = \mu$, $\omega = 1$, $B = 0$, $A = 1$, $a_1 = -\alpha + 1$, $z = q^\alpha$, $q^\lambda = h$, $\Gamma_q(\alpha)$ replaced by $G_q(\alpha)$ the operator (1) reduces to an operator due to W. A. Al-Salam and A. Verma [3].

In [6] the author obtained certain transformations involving these generalized fractional q-integral operators. The present paper deals with some new transformations of miscellaneous nature involving these operators.

2. Definitions and notations

The following definitions and notations will be used in this paper:

$$(q^\alpha)_n = (1-q^\alpha)(1-q^{\alpha+1})\dots(1-q^{\alpha+n-1}); \quad (q^\alpha)_0 = 1, \tag{3}$$

$$\Gamma_q(\alpha) = \frac{(1-q)_{\alpha-1}}{(1-q)^{\alpha-1}}, \quad (\alpha \neq 0, -1, -2, \dots), \tag{4}$$

$$e_q(x) = \sum_{r=0}^{\infty} \frac{x^r}{(q)_r} = \frac{1}{(1-x)_\infty}, \tag{5}$$

$$E_q(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r q^{r(r-1)/2}}{(q)_r} = (1-x)_\infty, \tag{6}$$

$$\int_0^x f(t)d(t;q) = x(1-q)\sum_{n=0}^{\infty} q^n f(xq^n), \tag{7}$$

$$\int_x^\infty f(t)d(t;q) = x(1-q)\sum_{n=1}^{\infty} q^{-n} f(xq^{-n}), \tag{8}$$

$$\int_0^\infty f(t)d(t;q) = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n), \tag{9}$$

$$
{}_A\Phi_B^{(q)}[(a);(b);x] \equiv {}_A\Phi_B[q^{(a)};q^{(b)};x] = \sum_{n=0}^{\infty} \frac{(q^{a_1})_n (q^{a_2})_n \dots (q^{a_A})_n x^n}{(q)_n (q^{b_1})_n (q^{b_2})_n \dots (q^{b_B})_n}, \quad |x|<1. \tag{10}
$$

$$
\Phi\left[\begin{matrix} \omega^\lambda z^\mu \\ x^\lambda y^\mu \end{matrix}\middle| \begin{matrix} & h^{(a)} & \\ h^{(b)} & ; & h^{(c)} \\ & h^{(d)} & \\ h^{(e)} & ; & h^{(f)} \end{matrix}\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\omega^{\lambda m} z^{\mu m} x^{\lambda n} y^{\mu m} [h^{(a)}]_{m+n} [h^{(b)}]_m [h^{(c)}]_n}{(h)_m (h)_n [h^{(d)}]_{m+n} [h^{(e)}]_m [h^{(f)}]_n} \tag{11}
$$

3. Transformations

This article deals with certain transformations of miscellaneous nature in the form of the following theorems:

Theorem 1. *If* $f(x) = \int_0^\infty x^{\alpha-1} y^{\beta-1}\, {}_r\Phi_{r-1}^{(h)}[(d_r);(e_{r-1});x^\lambda y^\lambda] g(y) d(y;q)$ *and* $\Psi(x) = I_q[(a);(b);\lambda,\omega;z,\lambda;\eta : f(x)]$, *where* $h=q^\lambda$, *then*

(12)

$$
\Psi(x) = \frac{x^{\alpha-1}}{(1-q^{\eta\lambda+\lambda-1})}\int_0^\infty y^{\beta-1} g(y)\Phi\left[\begin{matrix} \omega^\lambda z^\lambda \\ x^\lambda y^\lambda \end{matrix}\middle| \begin{matrix} & h^{\eta+1+\frac{1}{\lambda}(\alpha-1)} & \\ h^{(a)} & ; & h^{(d_r)} \\ & h^{(\eta+2+\frac{1}{\lambda}(\alpha-1)} & \\ h^{(b)} & ; & h^{(e_{r-1})} \end{matrix}\right] d(y;q),
$$

provided

(i) $|q|<1,\ |\omega z|<1,\ |x|<1,\ R\ell(\lambda)>0;$

(ii) $\sum_{r=-\infty}^{\infty} |q^{r(1+\beta-\eta\lambda-\lambda-\alpha)} g(q^r)|$ *is convergent*

and

(iii) $R\ell\lambda\gamma > R\ell(\eta\lambda+\lambda+\alpha-1) > 0$, *where* $\gamma = \min(d_1,d_2,\dots,d_r)$.

Theorem 2. *If* $f(x) = \int_0^\infty \frac{y^\beta}{x^\alpha}\, {}_r\Phi_{r-1}^{(h)}\left[\begin{matrix} (d_r); & \frac{y^\lambda}{x^\lambda} \\ (e_{r-1}); & \end{matrix}\right] g(y) d(y;q)$ *and*

$\Psi(x) = K_q[(a);(b);\lambda,\omega;z,\lambda;\eta : f(x)]$, *where* $h=q^\lambda$, *then*

(13)

$$
\Psi(x) = \frac{(q/x)^\alpha}{(1-q^{\eta\lambda+\lambda-1})}\int_0^\infty y^\beta g(y)\Phi^{(h)}\left[\begin{matrix} \omega^\lambda z^\lambda q^\lambda \\ q^\lambda y^\lambda/x^\lambda \end{matrix}\middle| \begin{matrix} & h^{\eta+1+\frac{1}{\lambda}(\alpha-1)} & \\ h^{(a)} & ; & h^{(d_r)} \\ & h^{(\eta+2+\frac{1}{\lambda}(\alpha-1)} & \\ h^{(b)} & ; & h^{(e_{r-1})} \end{matrix}\right] d(y;q),
$$

provided

(i) $|q| < 1$, $|\omega z| < 1$, $|x| < 1$, $R\ell(\lambda) > 0$;

(ii) $\sum_{r=-\infty}^{\infty} |q^{r(1+\beta-\eta\lambda-\lambda-\alpha)} g(q^r)|$ *is convergent*

and

(iii) $R\ell\lambda\gamma > R\ell(\eta\lambda + \lambda + \alpha - 1) > 0$, *where* $\gamma = \min(d_1, d_2, \ldots, d_r)$.

Particular cases of Theorems 1 and 2.
Setting $\lambda = \omega = 1$, $\alpha = \beta = c$ in theorems 1 and 2, we have

Corollary 1. *If*

$$f(x) = \int_0^\infty (xy)^{c-1} {}_r\Phi_{r-1}^{(q)} \left[\begin{matrix} (d_r); & xy \\ (e_{r-1}); & \end{matrix} \right] g(y) d(y;q)$$

and

$$\Psi(x) = I_q[(a);(b);z,\eta : f(x)],$$

then

(14)

$$\Psi(x) = \frac{x^{c-1}}{1-q^{\eta+c}} \int_0^\infty y^{c-1} g(y) \Phi^{(q)} \left[\begin{matrix} z \\ xy \end{matrix} \middle| \begin{matrix} (a) & \eta+c & (d_r) \\ & ; & \\ & \eta+c+1 & \\ (b) & ; & (e_{r-1}) \end{matrix} \right] d(y;q)$$

provided

(i) $|q| < 1$, $|z| < 1$, $|x| < 1$,

(ii) $\sum_{-\infty}^{\infty} |q^{-r\eta} g(q^r)|$ *is convergent*

and

(iii) $R\ell(\gamma) > R\ell(\eta + c) > 0$, *where* $\gamma = \min(d_1, d_2, \ldots, dr)$.

Corollary 2. *If* $f(x) = \int_0^\infty (\frac{y}{x})^c \, {}_r\Phi_{r-1}^{(q)}[(d_r;(e_{r-1}); \frac{y}{x}] g(y) \, d(y;q)$ *and* $\Psi(x) = K_q[(a);(b);z,\eta : f(x)]$, *then*

(15)

$$\Psi(x) = \frac{(q/x)^c}{1-q^{\eta+c}} \int_0^\infty y^c g(y) \Phi^{(q)} \left[\begin{matrix} zq \\ yq/x \end{matrix} \middle| \begin{matrix} (a) & \eta+c & (d_r) \\ & ; & \\ & \eta+c+1 & \\ (b) & ; & (e_{r-1} \end{matrix} \right] d(y;q)$$

provided

(i) $|q| < 1$, $|z| < 1$, $|x| < 1$,

(ii) $\sum_{-\infty}^{\infty} |q^{-r\eta} g(q^r)|$ *is convergent*

and

(iii) $R\ell(\gamma) > R\ell(\eta + c) > 0$, *where* $\gamma = \min(d_1, d_2, \ldots, dr)$.

Results (14) and (15) are due to M. Upadhyay [7].

Theorem 3. *If* $\Phi(x,y) = I_q[(a);(b);\lambda,\omega;z,\mu:[1-xyq^{\alpha}]_{-\alpha}\, f(x)]$ *and* $\Psi(x) = I_q[(a);(b);\lambda,\omega;z,\mu;\eta:h(x)]$ *then*

$$\Psi(x) = \frac{1}{(1-q)}\prod_q \begin{bmatrix} c, & \alpha - c \\ \alpha, & 1 \end{bmatrix} \int_0^\infty y^{c-1}\Phi(x,y)d(y;q)\,, \tag{16}$$

where

$$h(x) = \prod \begin{bmatrix} xq^c, & q^{1-c}/x; & q \\ x, & q/x & \end{bmatrix} f(x),$$

provided
(i) $R\ell(\alpha) > R\ell(c) > 0$, $|q| < 1$, $R\ell(\mu) > 0$, $|\omega^{\lambda}z^{\mu}| < 1$
and
(ii) the basic integrals for $\Phi(x,y)$ *and* $\Psi(x)$ *converge absolutely.*

Theorem 4. *If* $\Phi(x,y) = K_q[(a);(b);\lambda,\omega;z,\mu;\eta:[1-yq^{\alpha}/x]_{-\alpha}f(x)]$ *and* $\Psi(x) = K_q[(a);(b);\lambda,\omega;z,\mu;\eta:h(x)]$ *then*

$$\Psi(x) = \frac{1}{(1-q)}\prod_q \begin{bmatrix} \alpha - c, & c \\ c, & 1 \end{bmatrix} \int_0^\infty y^{c-1}\Phi(x,y)d(y;q), \tag{17}$$

where

$$h(x) = \prod \begin{bmatrix} xq^{1-c}, & q^c/x; & q \\ xq, & 1/x & \end{bmatrix} f(x),$$

provided
(i) $R\ell(\alpha) > R\ell(c) > 0$, $|q| < 1$, $R\ell(\mu) > 0$, $|\omega^{\lambda}z^{\mu}| < 1$
and
(ii) the q-integrals for $\Phi(x,y)$ *and* $\Psi(x)$ *converge absolutely.*

Particular cases of Theorems 3 and 4.

Case 1. Setting $\lambda = \omega = 1, \mu = 1$ in theorems 3 and 4, we obtain

Corollary 3. *If* $\Phi(x,y) = I_q[(a);(b);z,\eta:[1-xyq^{\alpha}]_{-\alpha}f(x)]$ *and* $\Psi(x) = I_q[(a);(b);z,\eta:h(x)]$, *then*

$$\Psi(x) = \frac{1}{1-q}\prod_q \begin{bmatrix} c, & \alpha - c \\ \alpha, & 1 \end{bmatrix} \int_0^\infty y^{c-1}\Phi(x,y)d(y;q), \tag{18}$$

where

$$h(x) = \prod \begin{bmatrix} xq^c, & q^{1-c}/x; & q \\ x, & q/x & \end{bmatrix} f(x),$$

provided

(i) $R\ell(\alpha) > R\ell(c) > 0$, $|z| < 1$, $|q| < 1$

and

(ii) *the q-integrals for* $\Phi(x,y)$ *and* $\Psi(x)$ *converge absolutely.*

Corollary 4. *If* $\Phi(x,y) = K_q[(a);(b);z,\eta : [1-yq^{\alpha}/x]_{-\alpha} f(x)]$ *and* $\Psi(x) = K_q[(a);(b);z,\eta : h(x)]$ *then*

$$\Psi(x) = \frac{1}{(1-q)} \prod_q \begin{bmatrix} \alpha-c, & c \\ \alpha, & 1 \end{bmatrix} \int_0^{\infty} y^{c-1}\, \Phi(x,y)\, d(y;q),$$

where

$$h(x) = \prod \begin{bmatrix} x & q^{1-c}, & q^{c}/x; & q \\ x & q, & 1/q & \end{bmatrix} f(x),$$

provided

(i) $R\ell(\alpha) > R\ell(c) > 0$, $|z| < 1$, $|q| < 1$

and

(ii) *the q-integrals for* $\Phi(x,y)$ *and* $\Psi(x)$ *converge absolutely.*

Case 2. Setting $B = 0$, $A = 1$, $a = -\alpha + 1$, $\lambda = 1$, $\mu = m$, $\omega = q^{\alpha-1}$ in theorems 3 and 4, also taking $z = q$ in theorem 3 and $z = 1$ in theorem 4, we obtain

Corollary 5. *If* $\Phi(x,y) = I^{\eta,\alpha}_{m,q}[1-xyq^{\alpha}]_{-\alpha}$ *and* $\Psi(x) = I^{\eta,\alpha}_{m,q} h(x)$, *then*

$$\Psi(x) = \frac{1}{1-q} \prod_q \begin{bmatrix} \gamma, & \alpha-\gamma \\ \alpha; & 1 \end{bmatrix} \int_0^{\infty} \Phi(x,y)\, d(y;q), \tag{20}$$

where

$$h(x) = \prod \begin{bmatrix} xq^{\gamma}, & q^{1-\gamma}/x; & q \\ x, & q/x & \end{bmatrix} f(x),$$

provided

(i) $R\ell(\alpha) > R\ell(\gamma) > 0$, $|z| < 1$, $|q| < 1$, *m is a positive integer*

and

(ii) *the basic integrqls for* $\Phi(x,y)$ *and* $\Psi(x)$ *converge absolutely.*

Corollary 6. *If* $\Phi(x,y) = K^{\eta,\alpha}_{m,q}[1-yq^{\alpha}/x]_{-\alpha} f(x)$ *and* $\Psi(x) = K^{\eta,\alpha}_{m,q} h(x)$ *then*

$$\Psi(x) = \frac{1}{1-q} \prod_q \begin{bmatrix} \alpha-\gamma, & \gamma \\ \alpha, & 1 \end{bmatrix} \int_0^{\infty} \Phi(x,y)\, d(y;q), \tag{21}$$

where

$$h(x) = \prod \begin{bmatrix} xq^{1-\gamma}, & q^{\gamma}/x; & q \\ xq, & 1/x & \end{bmatrix} f(x),$$

provided

(i) $R\ell(\alpha) > R\ell(\gamma) > 0$, $|z| < 1$, $|q| < 1$, *m is a positive integer and*

(ii) the q-integrals for $\Phi(x, y)$ *and* $\Psi(x)$ *converge absolutely.*

Results (18) and (19) are due to M. Upadhyay [7] and (20) and (21) are due to the author [4].

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Received 15.05.1992