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Disjointness Preserving Operator Semigroups on $C(X, \mathbb{C}^n)$

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1. Introduction

One-parameter semigroups of lattice homomorphisms on spaces $C(X)$ have been characterized by R. Derndinger- R. Nagel (see [2]). In particular they showed that the generator of such a semigroup is always the sum of a derivation and a multiplication operator.

In this paper we study the analogous problem for semigroups of disjointness preserving operators on the space $C(X, \mathbb{C}^n)$ of all continuous $\mathbb{C}^n$-valued functions on some compact space X. We show that the generator is again the sum of a derivation and a multiplication operator with values in the space $M_n(\mathbb{C})$ of all complex $n \times n$-matrices. Such operators occur, e.g., if we study non-autonomus linear Cauchy problems

$$\left(\frac{d}{dt} - A(t)\right) u(t) = 0,$$

for $u(t) \in \mathbb{C}^n, A(t) \in M_n(\mathbb{C})$ and $t \geq 0$ in the space $C([0, \infty], \mathbb{C}^n)$.

Another motivation for this investigation comes from topological dynamics where semigroups as studied below are called linear skew product flows and have been investigated in connection with the Oseledec Multiplicative Ergodic Theorem (e.g., see [5]).

2. Disjointness preserving operators on $C(X, \mathbb{C}^n)$

In the following we always take E to be the Banach space $C(X, \mathbb{C}^n)$ of all continuous $\mathbb{C}^n$-valued functions on some compact space X endowed with the norm

$$||f|| := \sup_{x \in X} |||f(x)|||$$

for $f \in E$ and some lattice norm $\| . \|$ on $\mathbb{C}^n$. If $n = 1$ we write $C(X)$ instead of $C(X, \mathbb{C})$ and occasionally we identify the space $C(X, \mathbb{C}^n)$ with $C(X) \otimes \mathbb{C}^n$ or with $C(Y)$ for Y the disjoint union of n copies of X.

On these spaces the following types of operators will be considered.

Definition 1.

(i) A bounded linear operator $T \in \mathcal{L}(E)$ is called disjointness preserving if for all functions $f, g \in E$ satisfying $\|f(x)\| . \|g(x)\| = 0 \quad \forall x \in X$ it follows that $\|Tf(x)\| . \|Tg(x)\| = 0 \quad \forall x \in X$.

(ii) Let $Q : X \mapsto M_n(\mathbb{C})$ be continuous. The operator $M \in \mathcal{L}(E)$ defined by

$$Mf(x) := Q(x) . f(x), \qquad x \in X, f \in E$$

is called the matrix multiplication operator corresponding to Q.

(iii) An (unbounded) linear operator δ with domain $D(\delta)$ in $C(X)$ is called a *-derivation if $D(\delta)$ is a *-subalgebra of $C(X)$ containing the constant function $1\!\!I$ and if

$$\delta(f\overline{g}) = f\overline{\delta g} + (\delta f)\overline{g}$$

for all $f, g \in D(\delta)$.

(iv) To a *-derivation δ on $C(X)$ we associate an operator δ_n on $C(X, \mathbb{C}^n)$ by

$$\delta_n(f_1, \ldots, f_n) := (\delta f_1, \ldots, \delta f_n)$$

for $f := (f_1, \ldots, f_n) \in C(X, \mathbb{C}^n)$ and all $f_i \in D(\delta)$.

Disjointness preserving operators on $C(X)$ have been first characterized by W. Arendt (see [1]). His result has then been extended to vector valued function spaces by J. E. Jamison - M. Rajagopalan (see [4]) and we restate their result for operators on $E = C(X, \mathbb{C}^n)$.

Propositon 2. *For an operator $T \in \mathcal{L}(E)$ the following assertions are equivalent.*

(a) T is disjointness preserving.

(b) There exist a continuous function $Q : X \mapsto M_n(\mathbb{C})$ and a map $\varphi : X \mapsto X$, continuous on $X \setminus \{x \in X : Q(x) = 0\}$, such that

$$Tf(x) = Q(x) . f(\varphi(x))$$

for $x \in X, f \in E$.

3. Characterization of disjointness preserving operator semigroups on $C(X, \mathbb{C}^n)$

We now consider strongly continuous one-parameter operator semigroups $(T(t))_{t \geq 0}$ on E. As usual its generator is $Af := \lim_{t \to 0} t^{-1}(T(t)f - f)$ for

$f \in D(A) := \{f \in E : \lim_{t\to 0} t^{-1}(T(t)f - f)$ exists$\}$. For the basic theory of one-parameter semigroups we refer to [3], [4] or [7].

Generalizing Theorem 2.2 from [2] we find that norm continuous semigroups of disjointness preserving operators are rather trivial.

Proposition 1. *For a strongly continuous semigroups $(T(t))_{t\geq 0}$ on E the following assertions are equivalent.*

(a) $(T(t))_{t\geq 0}$ is a disjointness preserving, norm continuous semigroup.

(b) The generator of $(T(t))_{t\geq 0}$ is a (bounded) matrix multiplication operator.

Proof. Assume that (a) holds. Since each $T(t)$ is a disjointness preserving operator for all $t \geq 0$, there exist by Propositions 2 continuous functions $Q_t : X \mapsto M_n(\mathbb{C})$ and mappings $\varphi_t : X \mapsto X$, which are continuous on $U_t := \{x \in X : Q_t(x) \neq 0\}$, such that $T(t)f(x) = Q_t(x) \cdot f(\varphi_t(x))$ for all $x \in X, f \in E$.

By the semigroup property of $(T(t))_{t\geq 0}$ it follows that $\varphi_{t+s} = \varphi_t \cdot \varphi_s$ for $0 \leq s, t$. Therefore it suffices to show that $\varphi_t = id_X$ for each t in a small interval $[0, t_0]$. Assume to the contrary that there exist $t \downarrow 0$ and $x_k \in X$ such that

$$\varphi_{t_k}(x_k) \neq x_k$$

for all $k \in \mathbb{N}$.

Then we choose functions $f_k \in C(X)$ satisfying $0 \leq f_k \leq 1\!\!I, f_k(x_k) = 0$ and $f_k(\varphi_{t_k}(x_k)) = 1$.

Denote by $1\!\!I_i$ the function $1\!\!I \otimes e_i$, where e_i is the i-th canonical basis vector in $\mathbb{C}^n$. The strong continuity of $(T(t))_{t\geq 0}$ implies that there exist $t_0 > 0$ such that

$$T(t)1\!\!I_i \geq \frac{1}{2} \cdot 1\!\!I_i$$

for $1 \leq i \leq n$ and $0 \leq t \leq t_0$. Since $T(t)1\!\!I_i(x) = Q_t(x)e_i$ for all $x \in X$ we conclude that all entries of $Q_t(x)$ are greater then or equal to zero, while the diagonal entries have lower bound $\frac{1}{2}$ for all $x \in X$ and $0 \leq t \leq t_0$. Therefore we obtain

$$\|T(t_k)(f_k \otimes e_i)(x_k) - (f_k \otimes e_i)(x_k)\| = \|Q_{t_k} \cdot f_k(\varphi_{t_k}(x_k)) \cdot e_i\| \geq \frac{1}{2}$$

for all $k \in \mathbb{N}$. This contradicts the norm continuity of $(T(t))_{t\geq 0}$, hence $\varphi_t = id_X$ for all $t \geq 0$ and

$$T(t)f(x) = Q_t(x) \cdot f(x),$$

which implies that the generator A is a matrix multiplication operator.

The implication $(b) \Rightarrow (a)$ follows since the semigroup generated by a bounded matrix multiplication is a semigroup of a matrix multiplication operators, hence disjointness preserving. ∎

For the characterization of the strongly continuous disjointness preserving semigroups we need one more concept (see [2] or [6], B-III).

Definition 2. A family of maps $(\varphi_t)_{t\geq 0}$ is called a flow on X, if
(i) $\varphi_t : X \mapsto X$ is continuous for every $t \geq 0$,
(ii) $\varphi_0 = id_X$,
(iii) $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for every $s, t \geq 0$.
The flow is continuous if the map $\varphi : \mathbb{R}_+ \times X \mapsto X$ defined by $\varphi(t,x) := \varphi_t(x)$ is continuous.

Remark 3. We refer to [6], Theorem 3.4 where it is shown how a continuous flow on X induces a strongly continuous semigroup on $C(X)$ whose generator is a derivation.

Definition 2 combined with the notions from Definition 1 allows to formulate our main result.

Theorem 4. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on $E = C(X, \mathbb{C}^n)$. Denote by $(A, D(A))$ its generator and assume that $1\!\!1 \otimes e_k \in D(A)$ for each canonical basis vector $e_k \in \mathbb{C}^n$. Then the following assertions are equivalent.*

(a) Each $T(t)$ is a disjointness preserving operator.

(b) There exists a matrix multiplication operator M on E and a $*$-derivation δ generating a strongly continuous semigroup on $C(X)$ such that $A = \delta_n + M$.

(c) There exists a continuous flow $(\varphi_t)_{t\geq 0}$ on X and a matrix multiplication operator M on E such that

$$T(t)f = \left(\lim_{n\to\infty}[V(\frac{t}{n})]^n\right) f \circ \varphi_t$$

where

$$V(\frac{t}{n}) := \exp\left(\int_0^{\frac{t}{n}} M(\varphi_s(x))ds\right)$$

for $f \in E, x \in X$ and $t \geq 0$.

Proof. "$(a) \Rightarrow (b)$": Since $(T(t))_{t\geq 0}$ is a disjointness preserving operator semigroup, there exist continuous functions $Q_t : X \to M_n(\mathbb{C})$ and mappings $\varphi_t : X \to X$, wich are continuous on $U_t := \{x \in X : Q_t(x) \neq 0\}$, such that

$$T(t)f = Q_t \cdot f \circ \varphi_t$$

for $f \in E$. By using the semigroup properties we obtain

$$(*) \qquad Q_{t+s}(x) \cdot f(\varphi_{t+s}(x)) = Q_t(x) \cdot Q_s(\varphi_t(x)) \cdot f(\varphi_t(\varphi_s(x)))$$

for all $f \in E, x \in X$ and $t, s \geq 0$. Let $Q_t(x) = (q_{t,ij}(x))_{n\times n}$. For $f = 1\!\!1 \otimes e_k$ we obtain from $(*)$

$$q_{t+s,ik}(x) = \sum_{j=1}^{n} q_{t,ij}(x) \cdot q_{s,jk}(\varphi_t(x))$$

for all $x \in X, t, s \geq 0$ and $1 \leq i, j \leq n$. This equality gives us the following identity

$$(**) \qquad Q_{t+s}(x) = Q_t(x) \cdot Q_s(\varphi_t(x))$$

for all $x \in X$ and $t, s \geq 0$. From $(*)$ and $(**)$ we obtain

$$(***) \qquad f(\varphi_{t+s}(x)) - f(\varphi_t(\varphi_s(x))) \in Ker Q_{t+s}(x)$$

for all $f \in E$ and $t, s \geq 0$. Next we show that $\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$ for all $x \in X$ and $t, s \geq 0$. Let $Q_{t+s}(x_0) = (q_{t+s,ij}(x_0))_{n \times n} \neq 0$ for a $x_0 \in X$. Then there exists a pair (i_0, j_0), $1 \leq i_0, j_0 \leq n$, such that $q_{t+s,i_0j_0}(x_0) \neq 0$. Assume that

$$x_1 := \varphi_{t+s}(x_0) \neq \varphi_t(\varphi_s(x_0)) =: x_2.$$

Let f be a continuous function on X satisfying

$$f(x) = \begin{cases} 0 & \text{for } x = x_1 \\ 1 & \text{for } x = x_2. \end{cases}$$

Then we have $f \otimes e_{j_0} \in E$ and

$$(\bullet) \qquad f(\varphi_{t+s}(x_0)) \cdot e_{j_0} - f(\varphi_t(\varphi_s(x_0))) \cdot e_{j_0} = e_{j_0} \neq 0.$$

And so

$$(\bullet\bullet) \qquad Q_{t+s}(x_0) \cdot e_{j_0} = q_{t+s,j_0}(x_0) \neq 0$$

for the j_0-th column of the matrix $Q_{t+s}(x_0)$. By the relations $(\bullet)$ and $(\bullet\bullet)$ we have shown

$$f(\varphi_{t+s}(x_0)) \cdot e_{j_0} - f(\varphi_t(\varphi_s(x_0))) \cdot e_{j_0} \notin Ker Q_{t+s}(x_0),$$

but this is a contradiction to $(***)$. Therefore $\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$ for all $x \in X$ and $t, s \geq 0$.

Let $S(t)g := g \circ \varphi_t$ for arbitrary $g \in C(X)$ and $t \geq 0$. Then $(S(t))_{t \geq 0}$ satisfies the semigroup properties. If course $g \otimes e_k \in E$ for all $g \in C(X)$ and $1 \leq k \leq n$. On the other hand,

$$\begin{aligned} \lim_{t \to 0} \|S(t)g - g\| &\leq \lim_{t \to 0} \|g \circ \varphi_t . e_k - g . e_k\| \\ &\leq \lim_{t \to 0} \|(Id - Q_t) g \circ \varphi_t . e_k\| + \lim_{t \to 0} \|T(t) g \otimes e_k - g \otimes e_k\| \\ &\leq \lim_{t \to 0} \sum_{k=1}^{n} \|g \circ \varphi_t (Id - Q_t)(1\!\!I \otimes e_k)\| + \lim_{t \to 0} \|T(t)g - g\| = 0 \end{aligned}$$

for all $g \in C(X)$. Therefore we obtain from the last inequality that $(S(t))_{t \geq 0}$ is strongly continuous and $S(t)1\!\!I = 1\!\!I$ for $1\!\!I \in C(X)$. By Theorem 3.4 in [6] the generator δ of $(S(t))_{t\geq 0}$ is a *-derivation. Now let $S_n(t)f := f \circ \varphi_t$ for $f = (f_1, \ldots, f_n) \in E$. Then $(S(t))_{t\geq 0}$ is a strongly continuous semigroup on E and its generator is the derivation δ_n on E. Let $M := \lim_{t\to 0} t_{-1}(Q_t - Id)$. Since $1\!\!I \otimes e_k \in D(A)$ for all $k, 1 \leq k \leq n$, this limit exists. Therefore

$$\begin{aligned} t^{-1}(T(t)f(x) - f(x)) = t^{-1}(Q_t(x)f(\varphi_t(x)) - Q_t(x)f(x)) + \\ t^{-1}(Q_t(x)f(x) - f(x)) = \\ = t^{-1}Q_t(x)(S_n(t)f(x) - f(x)) + t^{-1}(Q_t(x) - Id)f(x) \end{aligned}$$

for all $f \in E$ and $t \geq 0$. If we let t tends to 0 then we obtain

$$Af = \delta_n f + Mf.$$

The compactness of X gives us $M \in \mathcal{L}(E)$ and therefore the domain $D(\delta_n)$ is equal to the domain $D(\delta_n + M)$.

"$(b) \Rightarrow (c)$": Define the operator family $(W(t))_{t\geq 0}$ by

$$W(t)f(x) := \exp\left(\int_0^t M(\varphi_s(x))ds\right) \cdot f(\varphi_t(x))$$

for $f \in E, x \in X$ and $t \geq 0$. Then $(W(t))_{t\geq 0}$ satisfies all the assumptions of the Chernoff Product Formula (see [3], Theorem I.8.4) and it is easy to show that $W(0) = \delta_n + M$. Hence the limit of $\left[W\left(\frac{t}{n}\right)\right]^n$ as $n \to \infty$ coincides with the semigroup generated by $\delta_n + M$ and has the form stated in (c).

"$(c) \Rightarrow (a)$": Each $\left[V\left(\frac{t}{n}\right)\right]^n \circ \varphi_t$ is a disjointness preserving operator. Therefore the same holds for its limit with respect to the strong operator topology. ∎

Corollary 5. *Let $(T(t))_{t\geq 0}$ satisfy the assumptions and one of the conditions $(a) - (c)$ of Theorem 4. If the matrices $M(x), x \in X$, commute then the semigroup is given by*

$$T(t)f(x) := \exp\left(\int_0^t M(\varphi_s(x))ds\right) \cdot f(\varphi_t(x))$$

for $f \in E, x \in X$ and $t \geq 0$

Proof. Suppose that the condition (b) of Theorem 4 holds. Let us define

$$S(t)f(x) := \exp\left(\int_0^t M(\varphi_s(x))ds\right) \cdot f(\varphi_t(x))$$

for all$f \in E, x \in X$ and $t \geq 0$. Then it is easy to see that $(S(t))_{t\geq 0}$ is a strongly continuous ,disjointness preserving semigroup. Let $(B, D(B))$ be the generator of $(S(t))_{t\geq 0}$. Then

$$\frac{d}{dt}S(t)f_{|t=0} = \delta_n f + Mf$$

for $f \in D(\delta_n)$. Therefore $\delta_n + M \subset B$. Since $\delta_n + M$ is already a generator we conclude $\delta_n + M =$ and $S(t) = T(t)$ for all $t \geq 0$. ∎

We conclude this paper with an example originating from non-autonomous Cauchy problems as mentioned above. For simplicity we assume the matrix family $(A(t))_{t\geq 0}$ to be periodic.

Example 6. Take $X := \{z \in \mathbb{C} : |z| = 1\}$ and $E := C(X, \mathbb{C}^n)$. To the continuous flow

$$\varphi_t(z) := (e^{it}z), \qquad z \in X,$$

corresponds the derivation

$$\delta f(z) := z \cdot f'(z)$$

on $C(X)$. If M is the matrix multiplication operator on E corresponding to the continuous function

$$Q : X \to M_n(\mathbb{C}),$$

then the generator $\delta_n + M$ generates the disjointness preserving semigroup on E given by the formulas in Theorem 5(c).

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