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Normal Families of Meromorphic Functions

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After the introduction of weak continuous convergence, weak normality, and weak*-normality we study their properties and relations to continuous convergence and normality of sequences and families of functions, especially of meromorphic functions.

1. Introduction

If z_1 and z_2 are two points in the extended complex plane $\hat{C}$, the chordal distance (or chordal metric) of z_1 and z_2 is given by $\{p.\ 81, [1]\}$

$$\chi(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}}.$$

Let A be any set in the extended complex plane which is dense-in-itself and let $f_1(z), f_2(z), \ldots, f_n(z), \ldots$ be a sequence of complex functions, not necessarily analytic nor even continuous, defined on A. Let z_0 be a limit point of A, not necessarily belonging to A, and we consider a sequence of points $z_1, z_2, \ldots, z_n, \ldots$ from A converging to z_0. We form the sequence

$$\omega_n = f_n(z_n) \quad (n = 1, 2, 3, \ldots) \tag{1}$$

If the sequence (1) converges for every choice of the sequence $z_1, z_2, \ldots, z_n, \ldots$ as described above, then we say that the sequence of functions $f_n(z)$ is continuously convergent at z_0 $\{p.\ 174, [1]\}$.

It is also known $\{p.174, [1]\}$ that if a sequence $\{f_n(z)\}$ of functions is continuously convergent at z_0 then (I) the limit is independent of the choice of the sequence $\{z_n\}$ converging to z_0, and (II) every subsequence of $\{f_n(z)\}$ is continuously convergent at z_0.

Let now C_k be the circular disk defined by $\chi(z, z_0) < \frac{1}{k}$, $k = 1, 2, 3, \ldots$. Suppose that S_{nk} is the chordal oscillation of the function $f_n(z)$ on the set $A \cap C_k$ i.e., $S_{nk} = \sup\{\chi(f_n(z'), f_n(z'')) : z', z'' \in A \cap C_k\}$. We then form the numbers $\delta_k = \limsup_{n\to\infty} S_{nk}$, $k = 1, 2, \ldots$. Since $C_{(k+1)} \subset C_k$, it follows that for every n, $S_{n(k+1)} \leq S_{nk}$ and so $\delta_1 \geq \delta_2 \geq \delta_3 \geq \ldots$ holds. Hence the limit $\delta(z_0) = \lim_{k\to\infty} \delta_k$ exists, it called the limiting oscillation {p. 175, [1]} of the sequence of functions $\{f_n(z)\}$ at the point z_0.

The sequence $\{f_n(z)\}$ is said to be normal at z_0 if and only if $\delta(z_0) = 0$ {p. 178, [1]} and the set S of all points z_0 at which $\{f_n(z)\}$ is normal is called the normal kernel of the sequence $\{f_n(z)\}$ {p. 178, [1]}. Further $\{f_n(z)\}$ has a subsequence which is continuously convergent at every point of S {p. 178, *citer*1}.

A real-valued function $f(z)$ of a complex variable z defined in a domain $G \subset \hat{C}$ is said to be upper semi-continuous at a point $z_0 \in G$ if corresponding to ε (> 0), there exists a $\delta(> 0)$ such that

$$f(z) < f(z_0) + \varepsilon \qquad \text{for all } z \in G \cap \{z : \chi(z, z_0) < \delta\}.$$

It is observed {Theorem4, [2]} that the limiting oscilation function $\delta(z)$ is the upper semicontinuous at every point of its domain of definition.

Let $\mathcal{F}$ be a family of meromorphic functions, all defined on a region $G \subset \hat{C}$. A point z_0 in the interior of G is said to be a normal point {p. 182, [1]} of the family $\mathcal{F}$ if there is at least one positive number $\alpha < 1$ and at least one neighbourhood U of z_0 such that $\chi(f(z), f(z_0)) < \alpha$ for all $z \in U$ and for all $f(z) \in \mathcal{F}$. The collection S of all normal points of $\mathcal{F}$ is called the normal kernel of $\mathcal{F}$ {p. 182, [1]}. When $S = G$, we say that the family $\mathcal{F}$ is normal in G {p. 182, [1]}.

In this paper we introduce the notion of weak continuous convergence of a sequence of functions at a point and show that there are sequences which are weakly continuously convergent at a point but are not continuously convergent there at z_0. We also define the lower limiting oscillation of a sequence of functions at a point and establish a relation with the weak continuous convergence. Also we deal with the concept of weak* normality of a family of meromorphic functions at a point and we prove that this idea ultimately coincides with that of normality of the family.

Unless otherwise stated, sets are always subsets of $\hat{C}$, functions (generally complex valued) are defined on subsets of $\hat{C}$ and the distance between two points in $\hat{C}$ means their chordal distance.

2. Weak Continuous Convergence

We introduce the following definition.

Definition 1. A sequence $\{f_n(z)\}$ of functions is said to be weakly continuously convergent at a point z_0 of the closure of the common domain of definitions of $f_n(z)$, $n = 1, 2, \ldots$ if and only if at least one subsequence of $\{f_n(z)\}$ is continuously convergent at z_0.

Clearly a sequence which is continuously convergent at z_0 is weakly continuously convergent at z_0, but the following example shows that the converse is not true.

Example 1. Let $\{f_n(z)\}$ be defined in $0 < |z| < 1$ as follows:

$$f_n(z) = \begin{cases} \dfrac{1}{nz} & \text{if n is odd,} \\ z^n & \text{if n is even.} \end{cases}$$

The sequence $\{f_n(z)\}$ is not continuously convergent at $z = 0$, but it is weakly continuously convergent at $z = 0$, because the sequence $\{f_n(z) : n \text{ is even}\}$ is continuously convergent at $z = 0$.

3. Lower Limiting Oscillation and Weakly Normal Sequence

Definition 2. Let z_0 be a point of accumulation of the common domain of definition (a dense-in-itself set) of the sequence of functions $\{f_n(z)\}$. Let C_k be the circular discs defined by relation $\chi(z, z_0) < \frac{1}{k}$, $k = 1, 2, 3, \ldots$ and let S_{nk} be the chordal oscillation of the function $f_n(z)$ on $A \cap C_k$ i.e., $S_{nk} = \sup\{\chi(f_n(z'), f_n(z'')) : z', z'' \in A \cap C_k\}$. Let $\underline{\delta}_k = \liminf_{n\to\infty} S_{nk}$, $k = 1, 2, 3, \ldots$. Since we have $C_{(k+1)} \subset C_k$, it follows that for every n $S_{n(k+1)} \leq S_{nk}$ and so $\underline{\delta}_1 \geq \underline{\delta}_2 \geq \delta_3 \ldots$ holds. Hence the limit $\underline{\delta}(z_0) = \lim_{k\to\infty} \underline{\delta}_k$ exists, and we shall call it the lower limiting oscillation of the sequence of functions $\{f_n(z)\}$ at the point z_0.

Remark 1. Since the chordal distance of any two points does not exceed 1, it follows from the definitions that $0 \leq \underline{\delta}(z_0) \leq \delta(z_0) \leq 1$.

An immediate consequence of Definition 2 is the following theorem.

Theorem 1. *Let $\{f_n(z)\}$ be a sequence of functions each defined in a region G. Then $\underline{\delta}(z)$ is upper semi-continuous at every point of the closure of G.*

Proof. We shall show that if ε is an arbitrary positive number, then a neighbourhood of a point z_0 in the closure of G exists such that $\underline{\delta}(z) < \underline{\delta}(z_0) + \varepsilon$ for each point z in the common part of this neighbourhood and G.

If possible, suppose that this is not true. Then in every neighbourhood of z_0 there exists points $z' \in G$ such that $\underline{\delta}(z') \geq \underline{\delta}(z_0) + \varepsilon$. Let $N_1 : \chi(z, z_0) < \frac{1}{k}$, where k is a positive integer, be any neighbourhood of z_0. Let the positive integer k' be so chosen that $N_2 : \chi(z, z') < \frac{1}{k'}$, is contained in N_1. Since $\underline{\delta}(z') \geq \underline{\delta}(z_0) + \varepsilon$, from the definition of $\underline{\delta}(z)$ it follows that

$$\begin{aligned} \underline{\delta}_{k'} &= \liminf_{n\to\infty} S_{nk'} = \liminf_{n\to\infty} \sup_{z_1, z_2 \in N_2 \cap G} \chi(f_n(z_1), f_n(z_2)) \\ &\geq \underline{\delta}(z') \geq \underline{\delta}(z_0) + \varepsilon, \end{aligned}$$

which gives

$$\sup_{z_1,z_2\in N_2\cap G} \chi(f_n(z_1), f_n(z_2)) \geq \underline{\delta}(z_0) + \varepsilon/2$$

for all sufficiently large values of n. Since in each neighbourhood of z_0, we can always choose neighbourhood like N_2, this contradicts the definition of $\underline{\delta}(z_0)$. This proves the theorem. ■

The next theorem gives a relation between the lower limitimg oscillation of a sequence of functions at a point and the weak continuous convergence of the sequence at that point.

Theorem 2. *If $\underline{\delta}(z_0) > 0$, then $\{f_n(z)\}$ is not weakly continuously convergent at z_0.*

Proof. From the definition of $\underline{\delta}(z_0)$ it follows that

$$0 < \frac{\underline{\delta}(z_0)}{2} \leq \frac{\underline{\delta}_k}{2} < \underline{\delta}_k = \liminf_{n\to\infty} S_{nk} \qquad \text{for } k = 1, 2, 3, \ldots$$

which gives

$$\frac{\underline{\delta}(z_0)}{2} < S_{nk} \qquad \text{for } n \geq N_k, \text{ say, and } k = 1, 2, 3, \ldots$$

i.e.

$$\frac{\underline{\delta}(z_0)}{2} < \sup_{z',z''\in C_k\cap A} \chi(f_n(z'), f_n(z'')) \text{ for } n \geq N_k \text{ and } k = 1, 2, \ldots$$

So there exist points $z'_{nk}, z''_{nk} \in C_k \cap A$ such that

$$\frac{\underline{\delta}(z_0)}{2} < \chi(f_n(z'_{nk}, f_n(z''_{nk})) \text{ for } n \geq N_k \text{ and } k = 1, 2, 3, \ldots$$

Let $\{f_{n_k}(z)\}$ be any subsequence of $\{f_n(z)\}$. Then there exists a sequence of positive integers $\{p_k\}$ such that $\{f_{N_k+p_k}(z)\}$ is a subsequence of $\{f_{n_k}(z)\}$. Since for $k = 1, 2, 3, \ldots$ $z'_{(N_k+p_k),k}, z''_{(N_k+p_k),k} \in C_k \cap A$, it follows that

$$\lim_{k\to\infty} z'_{(N_k+p_k),k} = \lim_{k\to\infty} z''_{(N_k+p_k),k} = z_0.$$

We shall show that the sequences
$\{f_{N_k+p_k}(z'_{(N_k+p_k),k})\}$ and
$\{f_{N_k+p_k}(z''_{(N_k+p_k),k})\}$, if converge, converge to two different limits. Let

$$\lim_{k\to\infty} f_{N_k+p_k}(z'_{(N_k+p_k),k}) = \alpha \text{ and } \lim_{k\to\infty} f_{N_k+p_k}(z''_{(N_k+p_k),k}) = \beta.$$

Then for given ε, $0 < \varepsilon < \dfrac{\underline{\delta}(z_0)}{4}$, there exists a positive integer k_0 such that

$$\chi(f_{N_{k_0}+p_{k_0}}(z'_{(N_{k_0}+p_{k_0}),k_0}), \alpha) < \varepsilon \text{ and } \chi(f_{N_{k_0}+p_{k_0}}(z''_{(N_{k_0}+p_{k_0}),k_0}), \beta) < \varepsilon.$$

Then

$$\begin{aligned}\frac{\delta(z_0)}{2} &< \chi(f_{N_{k_0}+p_{k_0}}(z'_{(N_{k_0}+p_{k_0}),k_0}), f_{N_{k_0}+p_{k_0}}(z''_{(N_{k_0}+p_{k_0}),k_0}))\\ &\leq \chi(f_{N_{k_0}+p_{k_0}}(z'_{(N_{k_0}+p_{k_0}),k_0}), \alpha) + \chi(\alpha,\beta) + \\ &+ \chi(f_{N_{k_0}+p_{k_0}}(z''_{(N_{k_0}+p_{k_0}),k_0}), \beta)\\ &< \chi(\alpha,\beta) + 2\varepsilon,\end{aligned}$$

i.e., $\chi(\alpha,\beta) > \dfrac{\delta(z_0)}{2} - 2\varepsilon > 0$, which implies that $\alpha \neq \beta$. Therefore, the sequence $\{f_{N_{k_0}+p_{k_0}}(z)\}$ is not continuously convergent at z_0, and by our remark made in the introduction the sequence $\{f_{n_k}(z)\}$ is not continuously convergent at z_0. Since this is true for every subsequence of $\{f_n(z)\}$, it follows that the sequence $\{f_n(z)\}$ is not weakly continuously convergent at z_0. This proves the theorem. ∎

For a sequence of meromorphic functions we observe in the next theorem that the lower limiting oscillation has only two restricted values.

Theorem 3. *In the interior of their common domain G of definition, the lower limiting oscillation of any sequence $\{f_n(z)\}$ of meromorphic functions can assume only the values zero and unity.*

Proof. Let z_0 be a point of G at which the lower limiting oscillation $\underline{\delta}(z_0) < 1$. Let α be a positive number such that $\underline{\delta}(z_0) < \alpha < 1$. Since $\underline{\delta}(z_0) = \lim_{k\to\infty} \underline{\delta}_k$ and $\{\underline{\delta}_k\}$ is nonincreasing it follows that there is a neighbourhood C_k of z_0 such that $\underline{\delta}_k < \alpha$. Since $\underline{\delta}_k = \liminf_{n\to\infty} S_{nk}$, we have $S_{n_jk} \leq \alpha$ in $C_k \cap G$ for $j = 1,2,3,\ldots$. Therefore for all points in $C_k \cap G$, we obtain $\chi(f_{n_j}(z), f_{n_j}(z_0)) \leq \alpha$, $j = 1,2,3,\ldots$.

We set

$$g_{n_j}(z) = \begin{cases} \dfrac{f_{n_j}(z) - f_{n_j}(z_0)}{1 + \overline{f}_{n_j}(z_0) f_{n_j}(z)} & \text{if } f_{n_j}(z_0) \neq \infty \\ \dfrac{1}{f_{n_j}(z)} & \text{if } f_{n_j}(z_0) = \infty. \end{cases}$$

These functions $g_{n_j}(z)$ are meromorphic in $C_k \cap G$ and

$$\frac{|g_{n_j}(z)|}{\sqrt{1 + |g_{n_j}(z)|^2}} = \chi(g_{n_j}(z), 0) = \chi(f_{n_j}(z), f_{n_j}(z_0)) \leq \alpha,$$

which implies

$$|g_{n_j}(z)| \leq \frac{\alpha}{\sqrt{1-\alpha^2}} = M$$

say, for $j = 1,2,3,\ldots$ Thus we see that all the functions $g_{n_j}(z)$ are regular and uniformly bounded in $C_k \cap G$, and they all vanish at z_0. Let p be a positive integer. Applying Shwartz's lemma to the functions $\frac{1}{M} g_{n_j}(z)$, we see that there exists on the Riemann sphere a circular disc C_p with centre at z_0 in which $|g_{n_j}(z)| < \frac{1}{p}$ holds for $j = 1, 2, 3, \ldots$. At all the points of C_p and for $j = 1, 2, 3, \ldots$ we have

$$\chi(f_{n_j}(z), f_{n_j}(z_0)) = \frac{|g_{n_j}(z)|}{\sqrt{1 + |g_{n_j}(z)|^2}} \leq |g_{n_j}(z)| \leq \frac{1}{p}$$

so that $S_{n_j p} \leq \frac{2}{p}$. Therefore, it follows that $\underline{\delta}(z_0) \leq \underline{\delta}_p \leq \frac{2}{p}$, and since p is arbitrary, $\underline{\delta}(z_0) = 0$. This proves the theorem. ■

N o t e 1. It is known $\{p.\ 181, [1]\}$ that for a sequence of meromorphic functions the limiting oscillation at a point is either zero or unity. Certain connections between the limiting oscillation and the lower limiting oscillation will be shown in the latter part of the paper.

Definition 3. Let A be the common domain of the definition of the function of a sequence $\{f_n(z)\}$. We say that the sequence is weakly normal at the point z_0 of the closure of A if and only if $\underline{\delta}(z_0) = 0$. The set $\underline{S}$ of all points at which $\{f_n(z)\}$ is weakly normal is called the weak normal kernel of the sequence of functions.

Since $0 \leq \underline{\delta}(z_0) \leq \delta(z_0)$ at a point z_0 of the closure of A, it is clear that a sequence of functions, which is normal at a given point, is weakly normal at the same point.

Keeping in view some steps of proofs of Theorem 3, the following theorem on weakly normal sequence may be proved.

Theorem 4. *A sequence $\{f_n(z)\}$ of functions meromorphic in a region G is weakly normal at an interior point z_0 of G if and only if there exists at least one neighbourhood U of z_0 and at least one positive number $\alpha < 1$ such that for all points z of U and for a subsequence $\{f_{n_j}(z)\}$, $\chi(f_{n_j}(z), f_{n_j}(z_0)) < \alpha$ holds.*

In this case, we can assign to the arbitrary ε (> 0) a neighbourhood U_ε of z_0 such that for all points z of U_ε, we have $\chi(f_{n_j}(z), f_{n_j}(z_0)) < \varepsilon$, $j = 1, 2, 3, \ldots$ The set of those points of G at which the sequence $\{f_n(z)\}$ is weakly normal is then an open subset of G, unless it is empty.

N o t e 2. Using Theorem 1 we give in the following an alternative proof of the last statement of Theorem 4.

Let $\{f_n(z)\}$ be weakly normal at $z_0 \in G$ i.e., $\underline{\delta}(z_0) = 0$. Since $\underline{\delta}(z)$ is upper semi-continuous at z_0 ,there exists a neighbourhood U of z_0 such that $\underline{\delta}(z) < \underline{\delta}(z_0) + \frac{1}{2}$ for all $z \in U$. Since $\underline{\delta}(z_0) = 0$, it follows from Theorem 3 that $\underline{\delta}(z) = 0$ for all $z \in U$. Hence $\{f_n(z)\}$ is weakly normal at each point of U.

Next theorem gives a sufficient condition for a sequence of meromorphic functions to be weakly continuously convergent at a point of their common domain of definition.

Theorem 5. *Let $\{f_n(z)\}$ be a sequence of meromorphic functions and let G be their common domain of definition. If $\underline{\delta}(z_0) = 0$ for some $z_0 \in G$, then $\{f_n(z)\}$ converges weakly continuously at z_0.*

Proof. Since $\underline{\delta}(z_0) = 0$, by Theorem 4 there exist a positive number $\alpha < 1$ and a neighbourhood U of z_0 and a subsequence $\{f_{n_j}(z)\}$ of $\{f_n(z)\}$ such that $\chi(f_{n_j}(z), f_{n_j}(z_0)) < \alpha$ for all $z \in U$. By 181.7 {*p.* 181, [1]} we see that the limiting oscillation of the subsequence $\{f_{n_j}(z)\}$ is zero at z_0, and by § 179{*p.* 178, [1]} we can choose a subsequence of $\{f_{n_j}(z)\}$ which is continuously convergent at z_0. Therefore, the sequence $\{f_n(z)\}$ is weakly continuously convergent at z_0. This proves the theorem. ∎

Combining Theorem 2 and 5 we can state the following theorem.

Theorem 6. *A necessary and sufficient condition for a sequence $\{f_n(z)\}$ f meromorphic functions to be weakly continuously convergent at a point z_0 of their common domain of definition is $\underline{\delta}(z_0) = 0$.*

4. Weak* Normal Family

Definition 4. A family $\mathcal{F}$ of meromorphic functions is said to be weak* normal at an interior point z_0 of the common domain of definition G of the functions if for every infinite subfamily τ of $\mathcal{F}$ there exist

(i) a denumerable subfamily N of τ,

(ii) at least one positive number α_N (depending on the family N) such that $\alpha_N < 1$,

(iii) at least one neighbourhood U_N of z_0 (U_N depends on the family N) such that $\chi(f(z), f(z_0)) < \alpha_N$ for all $z \in N$ and for all $f(z) \in N$.

We shall show that the above family is identical with the normal family. Before proving this, we will obtain a connection between weak* normality and weak continuous convergence.

Theorem 7. *A family $\mathcal{F}$ of meromorphic functions is weak* normal at z_0 if and only if every sequence of functions from $\mathcal{F}$ is weakly continuously convergent at z_0.*

Proof. First we suppose that the family $\mathcal{F}$ is weak* normal at z_0. Let $\{f_n(z)\}$ be a sequence from $\mathcal{F}$. Then, by definition, there is a subsequence $\{f_{n_j}(z)\}$ of $\{f_n(z)\}$, a positive number $\alpha < 1$, a neighbourhood U of z_0 such that

$$\chi(f_{n_j}(z), f_{n_j}(z_0)) < \alpha \text{ for all } z \in U.$$

By 181.7 {p.181, [1]} the subsequence $\{f_{n_j}(z)\}$ is normal at z_0, and by §179 {p.178, [1]} we can select from $\{f_{n_j}(z)\}$ a subsequence which is continuously convergent at z_0. Therefore, $\{f_n(z)\}$ is weakly continuously convergent at z_0.

Next we suppose that every sequence of functions from $\mathcal{F}$ is weakly continuously convergent at z_0. If possible, suppose that the family $\mathcal{F}$ is not weak* normal at z_0. Then there exists an infinite subfamily τ of $\mathcal{F}$ such that for

$\alpha_N = 1 - \frac{1}{n}(n = 2, 3, \ldots)$ and for every neighbourhood $C_k : \chi(z, z_0) < \frac{1}{k}$ $(k = 1, 2, \ldots)$, the chordal oscillation of the functions $f(z) \in \tau$, exept possibly a finite number of functions (for each α_n), is not less then $1 - \frac{1}{n}$. So we can choose a sequence $\{f_n(z)\}$ of functions from τ such that the chordal oscillation of $f_n(z)$ on $C_k(k = 1, 2, \ldots)$ is not less then $1 - \frac{1}{n}$. Therefore, for this sequence $\underline{\delta}(z_0) = 1$ and so by Theorem 2 it is not weakly continuously convergent at z_0, which is a contradiction. This proves the theorem. ■

We now require the following known theorem {p. 183, [1]}.

Theorem 8. *A family $\mathcal{F}$ of meromorphic functions is normal in a region G if and only if from every sequence $f_1(z), f_2(z), f_3(z), \ldots$ of functions of the family, at least one subsequence $f_{k_1}(z), f_{k_2}(z), f_{k_3}(z), \ldots$ can be selected that converges continuously at every point of G.*

Combining Theorem 7 and 8 we get the following theorem.

Theorem 9. *A family $\mathcal{F}$ of meromorphic functions is normal at z_0 in the interior of their common domain of definition G if and only if the family is weak* normal at z_0.*

Remark 2. From Theorem 9 we see that the notion of weak* normality is identical with the notion of normality of a family of meromorphic functions. So Definition 4 can be treated as an alternative definition of normality at a point of a family of meromorphic functions, although the suppositions in Definition 4 are clearly weaker than the corresponding suppositions in the definition of normality of a family of meromorphic functions {p. 182, [1]}.

It is clear that $0 \leq \underline{\delta}(z_0) \leq \delta(z_0)$ and so if $\delta(z_0) = 0$ then $\underline{\delta}(z_0) = 0$. In the next theorem we prove the converse under certain additional supposition.

Theorem 10. *If the lower limiting oscillation of every subsequence of a sequence $\{f_n(z)\}$ of meromorphic functions is zero at a point z_0 belonging to the closure of their common domain of definition, then the limiting oscillation of the sequence $\{f_n(z)\}$ is also zero at z_0.*

Proof. By Theorem 4 the family $\{f_n(z)\}$ is weak* normal at z_0. So by Theorem 9 the family $\{f_n(z)\}$ is normal at z_0, and by 181.7 {p. 181, [1]} the limiting oscillation of the sequence $\{f_n(z)\}$ is zero at z_0. This proves the theorem. ■

Note 3. If a sequence is normal at z_0, then clearly every subsequence of it is normal at z_0 and so every subsequence of it is weakly normal at z_0. We can, therefore, restate Theorem 10 as follows:

A *sequence $\{f_n(z)\}$ of meromorphic functions is normal at a point z_0 if and only if every subsequence of it is weakly normal at z_0.*

Since for a sequence $\{f_n(z)\}$ of functions we get $0 \leq \underline{\delta}(z_0) \leq \delta(z_0) \leq 1$, it follows that $\underline{\delta}(z_0) = 1$ implies $\delta(z_0) = 1$. However for the converse part we prove the following theorem.

Theorem 11. *If the limiting oscillation of every subsequence of a sequence* $\{f_n(z)\}$ *of meromorphic functions is unity at a point* z_0 *belonging to the closure of their common domain of definition, then the lower limiting oscillation of the sequence* $\{f_n(z)\}$ *is also unity at* z_0 *.*

Proof. From the given condition it follows, in view of 181.7 {*p.* 181, [1]} that no subfamily of $\{f_n(z)\}$ is normal at z_0 . So by Theorem 8 no subsequence of $\{f_n(z)\}$ is weakly continuously convergent at z_0 , and in view of Theorem 6, the lower limiting oscillation of the sequence $\{f_n(z)\}$ is positive at z_0 . Since the functions $f_n(z)$ are meromorphic, by Theorem 3 the lower limiting oscillation of $\{f_n(z)\}$ is unity at z_0 . This proves the theorem. ■

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