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On the Space of a Class of Functions of Several Complex Variables Represented by Dirichlet Series

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1. Introduction

Consider the function of two complex variables represented by Dirichlet series

$$f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{mn} e^{s_1 \lambda_m + s_2 \mu_n} \tag{1}$$

where

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_m < \ldots \to \infty ,$$

$$0 < \mu_1 < \mu_2 < \ldots < \mu_n < \ldots \to \infty ,$$

$a_{mn} \in C$ and $s_1, s_2) \in C^2$, C being the field of complex numbers.

If (1) converges for $s_1 = \sigma_1 + it_1$, $s_2 = \sigma_2 + it_2$, then the series converges in the domain

$$D' = \{(\sigma_1' + it_1,\ \sigma_2' + it_2) \in C^2 \ :\ \sigma_1' \leq \sigma_1,\ \sigma_2' \leq \sigma_2,\ -\infty < t_1, t_2 < \infty\} ;$$

σ_1 and σ_2 are called the abscissae of convergence.

Let

$$\alpha(s_1, s_2) = \sum_{m,n=1}^{\infty} \alpha_{mn} e^{s_1 \lambda_m + s_2 \mu_n}$$

be a fixed Dirichlet series where none of the coefficients α_{mn} is equal to zero and λ_m's and μ_n's satisfy the condition

$$\lim_{m \to \infty} \frac{\log m}{\lambda_m} = 0 = \lim_{n \to \infty} \frac{\log n}{\mu_n} \tag{2}$$

From (2) it follows that its abscissae of absolute convergence coincides with its abscissae of ordinary convergence [2]. Let $c_1, c_2 > 0$ and suppose that the maximal abscissae of convergence of the series for $\alpha(s_1, s_2)$ are greater than or equal to c_1 and c_2 respectively, i.e., the series for α converges in the domain

$$D = \{(\sigma_1 + it_1, \sigma_2 + it_2) \in C^2 \ :$$

$$\sigma_1 < c_1,\ \sigma_2 < c_2,\ -\infty < t_1, t_2 < \infty\} \ .$$

Let Λ represent the class of all functions $f = f(s_1, s_2)$ represented by (1) for which the exponents λ_m's and μ_n's satisfy the same conditions (2) as that of α and let further

$$\lim_{m+n\to\infty} \left|\frac{a_{mn}}{\alpha_{mn}}\right| = 0 \ . \tag{3}$$

It is easily seen that if the maximal abscissae of convergence of α are greater than or equal to c_1, c_2 respectively, then every element of Λ has its maximal abscissae of convergence greater than or equal to c_1, c_2 respectively. It is clear that Λ does not include all functions whose maximal abscissae of convergence are greater than or equal to c_1, c_2 respectively.

In this paper, for brevity, we consider functions of two variables only, though the results obtained here can easily be extended to any finite number of variables.

2. Now define the following operations, namely addition(+), scalar multiplication (.), multiplication (o) and $\|\,.\,\|$ for the elements of Λ in the following way:

(i) $$(f+g)(s_1, s_2) = \sum_{m,n=1}^{\infty} (a_{mn} + b_{mn}) e^{s_1\lambda_m + s_2\mu_n},$$

(ii) $$(c.f)(s_1, s_2) = \sum_{m,n=1}^{\infty} (c a_{mn}) e^{s_1\lambda_m + s_2\mu_n},\ c \text{ being a complex number},$$

(iii) $$(f \circ g)(s_1, s_2) = \sum_{m,n=1}^{\infty} \frac{a_{mn} b_{mn}}{\alpha_{mn}} e^{s_1\lambda_m + s_2\mu_n}$$

and

(iv) $\|f\| = \sup_{m,n\geq 1} \left|\frac{a_{mn}}{\alpha_{mn}}\right|$,

where $f, g \in \Lambda$ such that

$$f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{mn} e^{s_1\lambda_m + s_2\mu_n},$$

$$g(s_1, s_2) = \sum_{m,n=1}^{\infty} b_{mn} e^{s_1\lambda_m + s_2\mu_n}.$$

It is easily verified that under these operations, Λ becomes a normed linear space over C.

Further it will be shown that Λ is a nonuniformly convex Banach space. A characterisation of linear functionals on Λ has also been obtained. This generalises some of the results obtained in [3] for analytic Dirichlet functions of one complex variable.

Theorem 1. *Λ is a nonuniformly convex Banach space which is separable also.*

(A normed linear space X is called uniformly convex if, to each $\epsilon > 0$, there corresponds $\delta(\epsilon) > 0$ such that $\|x+y\| \leq 2(1-\delta(\epsilon))$ when $\|x\| \leq 1, \|y\| \leq 1$ and $\|x-y\| \geq \epsilon$. This concept is due to J.A. Clarkson [1]).

Proof. (i) Λ is a Banach space.

Let $\{f_p\}$, where

$$f_p(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{mn}^{(p)} e^{s_1\lambda_m + s_2\mu_n} \qquad (p = 1, 2, \ldots)$$

be a Cauchy sequence in Λ so that

$$\left|\frac{a_{mn}^{(p)}}{\alpha_{mn}}\right| \to 0 \text{ for each } p \text{ as } m+n \to \infty \text{ (by (3))} .$$

Since $\{f_p\}$ is a Cauchy sequence in Λ , for a given $\epsilon > 0$, there exists a positive integer p_0 such that

$$\|f_p - f_q\| < \epsilon, \qquad \text{for } p, q \geq p_0.$$

This gives

$$\sup_{m,n\geq 1} \left|\frac{a_{mn}^{(p)} - a_{mn}^{(q)}}{\alpha_{mn}}\right| < \epsilon$$

for $p, q \geq p_0$.

Therefore,

$$\left|\frac{a_{mn}^{(p)} - a_{mn}^{(q)}}{\alpha_{mn}}\right| < \epsilon \tag{4}$$

for $p, q \geq p_0$ and for all $m, n \geq 1$.

This relation shows that $\left\{\frac{a_{mn}^{(p)}}{\alpha_{mn}}\right\}$ is a Cauchy sequence in C for each fixed m, n and so converges to $\frac{a_{mn}}{\alpha_{mn}}$ (say) $\in C$ as $p \to \infty$.

Hence we obtain

$$\left|\frac{a_{mn}}{\alpha_{mn}}\right| = \left|\frac{a_{mn}}{\alpha_{mn}} - \frac{a_{mn}^{(p)}}{\alpha_{mn}} + \frac{a_{mn}^{(p)}}{\alpha_{mn}}\right| \leq \left|\frac{a_{mn}}{\alpha_{mn}} - \frac{a_{mn}^{(p)}}{\alpha_{mn}}\right| + \left|\frac{a_{mn}^{(p)}}{\alpha_{mn}}\right| \rightarrow 0 \text{ as } m+n \rightarrow \infty.$$

This implies that f, where $f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{mn} e^{s_1\lambda_m + s_2\mu_n}$ belongs to Λ . Moreover,

$$\|f_p - f\| = \sup_{m,n\geq 1} \left|\frac{a_{mn}^{(p)} - a_{mn}}{\alpha_{mn}}\right| \rightarrow 0 \text{ as } p \rightarrow \infty, \text{ by (4)}.$$

Hence Λ is complete and Λ is therefore a Banach space.

(ii) Λ is not uniformly convex.

Let f, g be defined by

$$f(s_1, s_2) = \alpha_{m_0 n_0} e^{s_1\lambda_{m_0} + s_2\mu_{n_0}}$$

and

$$g(s_1, s_2) = \alpha_{m_0 n_0} e^{s_1\lambda_{m_0} + s_2\mu_{n_0}} + \alpha_{m_k n_k} e^{s_1\lambda_{m_k} + s_2\mu_{n_k}}$$

where (m_0, n_0) and (m_k, n_k) are pairs of fixed integers.

Evidently, $f, g \in \Lambda$ and $\|f\| = 1 = \|g\|$, $\|f - g\| = 1 \geq \epsilon$, but $\|f + g\| = 2 \nleq 2 - 2\delta(\epsilon)$ for any positive $\delta(\epsilon)$ proving that Λ is not uniformely convex.

(iii) Λ is separable.

To show this, let Λ_0 be the set of all functions $\sum_{p=1}^{m}\sum_{q=1}^{n} b_{pq} e^{s_1\lambda_p + s_2\mu_q}$ with rational complex coefficients, where m and n are finite, positive integers.

It is easily seen that the set Λ_0 is enumerable. Also, it is everywhere dence in Λ. To see this, let $\epsilon > 0$ be given and $f \in \Lambda$, where $f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{mn} e^{s_1\lambda_m + s_2\mu_n}$ so that $\left|\frac{a_{mn}}{\alpha_{mn}}\right| \rightarrow 0$ as $m+n \rightarrow \infty$ which gives that $\left|\frac{a_{mn}}{\alpha_{mn}}\right| <$ $\epsilon/2$ for $m + n > N_0$ which then implies that

$$\sup_{\substack{m,n \\ m+n>N_0}} \left|\frac{a_{mn}}{\alpha_{mn}}\right| \leq \epsilon/2 .$$

Let $g \in \Lambda$ be defined as follows:

$$g(s_1, s_2) = \sum_{m,n=1}^{\infty} b_{mn} e^{s_1\lambda_m + s_2\mu_n}$$

where

$$b_{mn} = 0 \text{ for } m + n > N_0$$

and

$$\left| \frac{a_{mn} - b_{mn}}{\alpha_{mn}} \right| < \epsilon/2$$

for m, n with $m + n \leq N_0$.

Then

$$\|f - g\| \leq \sup_{\substack{m,n \\ m+n \leq N_0}} \left| \frac{a_{mn} - b_{mn}}{\alpha_{mn}} \right| + \sup_{\substack{m,n \\ m+n > N_0}} \left| \frac{a_{mn}}{\alpha_{mn}} \right| \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon \, .$$

Thus Λ is separable and this proves the theorem.

From our definitions and also in view of Theorem 1, we may now state the following

Theorem 2. *Λ is a comutative Banach algebra.*

3. Linear Functionals on Λ

In this section we give the characterisation of bounded linear functionals on Λ.

Theorem 3. *Every bounded linear functional F defined for $f \in \Lambda$ is of the form*

$$F(f) = \sum_{m,n=1}^{\infty} (S) a_{mn} t_{mn} \, ,$$

where

$$f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{mn} e^{s_1 \lambda_m + s_2 \mu_n} \, ,$$

$$\sum_{m,n=1}^{\infty} |\alpha_{mn} t_{mn}| < \infty \, ,$$

and $\sum_{m,n=1}^{\infty} (S) a_{mn} t_{mn}$ stands for some convergent rearrangement of the terms of the series $\sum_{m,n=1}^{\infty} a_{mn} t_{mn}$ as a simple series.

To prove this theorem, we require the following

Lemma. *$f_N \to f (N \to \infty)$, where*

$$f_N(s_1, s_2) = \sum_{m,n=1}^{m+n=N} a_{mn} e^{s_1\lambda_m + s_2\mu_n} .$$

and

$$f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{mn} e^{s_1\lambda_m + s_2\mu_n} ,$$

for (s_1, s_2) belonging to the domain of convergence of $\alpha(s_1, s_2)$ if

$$|\frac{a_{mn}}{\alpha_{mn}}| \to 0 \text{ as } m + n \to \infty \text{ i.e. if } f \in \Lambda .$$

Proof of the Lemma. Let $f \in \Lambda$, where

$$f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{mn} e^{s_1\lambda_m + s_2\mu_n} ,$$

so that

$$|\frac{a_{mn}}{\alpha_{mn}}| \to 0 \text{ as } m + n \to \infty .$$

Hence

$$\|f - f_N\| = \sup_{\substack{m,n \\ m+n>N}} \left|\frac{a_{mn}}{\alpha_{mn}}\right| \to 0 \text{ as } N \to \infty .$$

Proof of the Theorem. Let F be defined for $f \in \Lambda$ as follows:

$$F(f) = \sum_{m,n=1}^{\infty} (S) a_{mn} t_{mn},$$

where

$$f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{mn} e^{s_1\lambda_m + s_2\mu_n} ,$$

and

$$\sum_{m,n=1}^{\infty} |\alpha_{mn} t_{mn}| < \infty .$$

Since

$$\sum_{m,n=1}^{\infty} |a_{mn} t_{mn}| \leq \sup_{m,n} |\frac{a_{mn}}{\alpha_{mn}}| \sum_{m,n=1}^{\infty} |\alpha_{mn} t_{mn}|$$

$$= \|f\| \sum_{m,n=1}^{\infty} |\alpha_{mn} t_{mn}| < \infty ,$$

F is a well-defined functional on Λ, and it is linear. Furthermore,

$$|F(f)| \leq \sum_{m,n=1}^{\infty} |a_{mn}t_{mn}| \leq \|f\| \sum_{m,n=1}^{\infty} |a_{mn}t_{mn}|$$

whence

$$\|F\| \leq \sum_{m,n=1}^{\infty} |a_{mn}t_{mn}| .$$

Thus F is a bounded linear functional on Λ. Consequently $F \in \Lambda^*$, the dual space of Λ.

Conversely, let $F \in \Lambda^*$ be defined as

$$F(\delta_{mn}) = t_{mn}, \text{ where } \delta_{mn} \in \Lambda \text{ is given by}$$

$$\delta_{mn}(s_1, s_2) = e^{s_1\lambda_m + s_2\mu_n} , m, n \geq 1 .$$

Then for any $f \in \Lambda$, where

$$\begin{aligned} f(s_1, s_2) &= \sum_{m,n=1}^{\infty} a_{mn}e^{s_1\lambda_m+s_2\mu_n} , \\ &= \sum_{m,n=1}^{\infty} a_{mn}\delta_{mn}(s_1, s_2) , \\ [1ex]F(f) &= F(\lim_{N\to\infty} f_N) \text{ (by the lemma)} \\ &= F(\lim_{N\to\infty} \sum_{m,n=1}^{m+n=N} a_{mn}\delta_{mn}) \\ &= \lim_{N\to\infty} [\sum_{m,n=1}^{m+n=N} a_{mn}F(\delta_{mn})] \\ &= \lim_{N\to\infty} [\sum_{m,n=1}^{m+n=N} a_{mn}t_{mn}] . \end{aligned}$$

So, for every $f \in \Lambda$, the series $\sum_{m,n=1}^{\infty} a_{mn}t_{mn}$ converges for some arrangement as a simple series and

$$F(f) = \sum_{m,n=1}^{\infty} (S)a_{mn}t_{mn} .$$

It will now be shown that

$$\sum_{m,n=1}^{\infty} |\alpha_{mn} t_{mn}| \leq \|F\|$$

so that

$$\sum_{m,n=1}^{\infty} |\alpha_{mn} t_{mn}| < \infty .$$

Take any $k \geq 2$ and let

$$a_{mn} = \begin{cases} |\alpha_{mn}| \operatorname{sgn}(t_{mn}) & \text{for } m+n \leq k \\ 0 & \text{for } m+n > k . \end{cases}$$

Define f by

$$f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{mn} e^{s_1 \lambda_m + s_2 \mu_n} ,$$

Evidently, $f \in \Lambda$ and $\|f\| = 1$.Therefore

$$|F(f)| = | \sum_{m,n=1}^{m+n=K} |\alpha_{mn}| \operatorname{sgn}(t_{mn}) F(\delta_{mn})|$$
$$= \sum_{m,n=1}^{m+n=K} |\alpha_{mn} t_{mn}| \left(\operatorname{sgn}(t_{mn}) = \frac{|t_{mn}|}{t_{mn}} \right) .$$

But $|F(f)| \leq \|F\| . \|f\| = \|F\|$ (since $\|f\| = 1$)so that

$$\sum_{m,n=1}^{m+n=K} |\alpha_{mn} t_{mn}| \leq \|F\|$$

and consequently

$$\sum_{m,n=1}^{\infty} |\alpha_{mn} t_{mn}| \leq \|F\| ,$$

and this prove the theorem.

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