Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol. 7, 1993, Fasc. 2

Remarks on the Homologically Trivial Spaces*

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A new proof of a sufficient condition for a homologically trivial space to be a Stein space is given. Some examples of homologically trivial spaces are considered.

1. Introduction

A detailed study of the theory of homology with coefficients in an analytic sheaf on complex spaces is made in the book of V. D. Golovin [1], where duality theorems are proved and some other applications are given.

The homologically trivial spaces are introduced and studied in [2], [3]. These are the countable complex spaces X with trivial homology spaces with compact support $H_k^c(X;\mathcal{F})$ for each $k \neq 0$ and each coherent sheaf $\mathcal{F}$ on X. Here a new simple proof of the following theorem, solving the Levi problem for this class of spaces will be given.

Theorem A. (cf.[2],[3]). *A homologically trivial complex space X is a Stein space if and only if the topological vector space $H_0^c(X;\mathcal{O}_X)$ is Hausdorff separable, where $\mathcal{O}_X$ is the structure sheaf on X.*

Some examples will be considered as well.

2. Preliminaries

The following statements, that will be needed in the proof of the Theorem A, will be given below.

* Partially supported by the Ministry of Education and Science of Bulgaria, grant No. 420.

Proposition B. (cf. [2],[3]). *A countable complex space X is homologically trivial if and only if the cohomology spaces $H^k(X;\mathcal{F})$ are trivial for each $k \geq 2$ and each coherent sheaf $\mathcal{F}$ on X and the cohomology space $H^1(X;\mathcal{F})$ has a trivial topology.*

Theorem C. (cf. [1], p. 70). *For each exact short sequence of analytic sheaves on the complex space X*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

homomorphisms $\partial : H^c_{k-1}(X;\mathcal{F}') \to H^c_k(X;\mathcal{F}'')$ are defined in a way that the sequence

$$\ldots \to H^c_k(X;\mathcal{F}'') \to H^c_k(X;\mathcal{F}) \to H(X;\mathcal{F}') \xrightarrow{\partial} H^c_{k-1}(X;\mathcal{F}'') \to \ldots$$
$$\ldots \to H^c_0(X;\mathcal{F}'') \to H^c_0(X;\mathcal{F}) \to H^c_0(X;\mathcal{F}') \to 0$$

should be an exact sequence of homomorphisms of topological vector spaces.

Theorem D. (cf. [1], p.91). *Let X be a countable complex space and let $\mathcal{F}$ be a coherent analytic sheaf on X. Then the topological vector space $\tilde{H}^c_k(X;\mathcal{F})$ is naturally isomorphic to the strongly dual space of the topological vector space $\tilde{H}^k(X;\mathcal{F})$, where the sign $\tilde{}$ stays for the factorization by the closure of the zero in the corresponding topological vector space and $H^k(X;\mathcal{F})$ is a space of the cohomology on X with coefficients in the sheaf $\mathcal{F}$:*

$$\{\tilde{H}^k(X;\mathcal{F})\}' = \tilde{H}^c_k(X;\mathcal{F}).$$

Corollary E. (cf. [1], p.92). *The topological vector space $H^k(X;\mathcal{F})$ is Hausdorff separable if and only if the topological vector space $H^c_{k-1}(X;\mathcal{F})$ is Hausdorff separable.*

Theorem F. (cf. [4], Chapter 5, §4.2, Theorem 3). *A countable complex space X is a Stein space if and only if the spaces of cohomology $H^1(X;\mathcal{I})$ are trivial for each coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$.*

3. Proof of theorem A

First note that each Stein space is a homologically trivial space because of Theorem B of Cartan and Proposition B above. To prove the sufficiency of the condition first we need to prove the following general statement.

Lemma. *Let A and B be topological vector spaces, $B \subset A$ and the factor space A/B is equipped with the natural factor topology. If the topological vector spaces B and A/B are Hausdorff separable, then the topological vector space A is also Hausdorff separable.*

Proof. Indeed, if two different points a and b from A posess two distinguished classes in the factor space A/B, then the inverse image of the open sets in A/B which separate their factor classes, separate them in A. If the points a and b from A, $a \neq b$, belong to the same factor class in A/B, then $a-b \in B$ and there exists an open neighborhood U of the origin in B which does not contain the point $a-b$, as the space B is Hausdorff separable. Since B is a subspace of A there exists an open neighborhood V in A such that $U = V \cap B$ and obviously $a - b \notin V$. Then if V' is an open set in A such that $V' + V' \subset V$, the open sets $a + V'$ and $b + V'$ in A will separate the points a and b in A. The lemma is proved.

Now to prove the Theorem A we shall show that the sufficient condition for the complex space to be a Stein space in the Theorem F above holds for each homologically trivial space X with separable space $H_0^c(X;\mathcal{O}_X)$.

First, we shall prove the separability of the space $H^1(X;\mathcal{F})$ for each coherent sheaf $\mathcal{F}$ on the homologically trivial space X. For this purpose we construct the exact short sequence of sheaves

$$0 \to \mathcal{O}_X^p \to \mathcal{O}_X^{p+1} \to \mathcal{O}_X \to 0$$

for each natural number p. By Theorem C above we obtain the exact sequence

$$H_1^c(X;\mathcal{O}_X^p) \xrightarrow{\partial} H_0^c(X;\mathcal{O}_X) \to H_0^c(X;\mathcal{O}_X^{p+1}) \to H_0^c(X;\mathcal{O}_X^p) \to 0$$

in which $H_1^c(X;\mathcal{O}_X) = 0$, as the space X is homologically trivial. Since we have assumed that the topological vector space $H_0^c(X;\mathcal{O}_X)$ is Hausdorff separable, using the above lemma with the help of induction on the natural number p, we obtain that the topological vector spaces $H_0^c(X;\mathcal{O}_X^p)$ are Hausdorff separable for each natural number p.

Let now F be a coherent sheaf on the homologically trivial space X. We consider the exact sequence of sheaves homomorphisms

$$\mathcal{O}_X^q \to \mathcal{O}_X^p \xrightarrow{\lambda} \mathcal{F} \to 0$$

which always exists for the coherent sheaf F on the complex space X. This sequence induce a short exact sequence of the kind

$$0 \to \operatorname{Ker}\lambda \to \mathcal{O}_X^p \xrightarrow{\lambda} \mathcal{F} \to 0$$

in which two of sheaves are coherent. By the Three-lemma there follows that the third sheaf is again a coherent sheaf. Then we construct the exact sequence in Theorem C for this short sequence of sheaves. Its last terms are the following ones:

$$H_1^c(X;\mathrm{Ker}\lambda) \xrightarrow{\partial} H_0^c(X;\mathcal{F}) \to H_0^c(X;\mathcal{O}_X^p) \to H_0^c(X;\mathrm{Ker}\lambda) \to 0.$$

Since the sheaf $\mathrm{Ker}\lambda$ is a coherent sheaf and X is homologically trivial space, by definition $H_1^c(X;\mathrm{Ker}\lambda) = 0$ and the topological vector space $H_0^c(X;\mathcal{F})$ is a subspace of the topological vector space $H_0^c(X;\mathcal{O}_X^p)$. But we have proved that these spaces are Hausdorff separable. So the topological vector space $H_0^c(X;\mathcal{F})$ is Hausdorff separable. Then according to Corollary E the topological vector space $H^1(X;\mathcal{F})$ will be Hausdorff separable.

But according to Proposition B the topology of $H^1(X;\mathcal{F})$ is trivial for each coherent sheaf $\mathcal{F}$ on the homologically trivial space X. So $H^1(X;\mathcal{F}) = 0$ when the topological vector space $H_0^c(X;\mathcal{O}_X)$ is Hausdorff separable. By Theorem F it follows that X is a Stein space in this case. Theorem A is proved.

4. Some examples of homologically trivial spaces

As we have seen, each Stein space is a homologically trivial space. But there exist homologically trivial spaces, which are not Stein spaces. According to Theorem A such are all countable complex spaces so that $H_k^c(X;\mathcal{F}) = 0$ for each $k \neq 0$ and each coherent sheaf $\mathcal{F}$ on X and with not Hausdorff separable vector space $H_0^c(X;\mathcal{O}_X)$.

One class of examples is obtained by a Theorem stated in [2] and proved in [3], namely these are the complex spaces, which are unions of increasing sequences of homologically trivial open sets. In particular, the complex spaces which are unions of increasing sequences of Stein open sets give examples of homologically trivial spaces and there exists an example of such an union, which is not a Stein space (cf. [5]).

One more class of examples of homologically trivial spaces can be constructed generalizing a construction in [2], [3]. Namely, if X is a Stein space, Y is a subset of X which is also a Stein space and $Y \times Y$ is relatively Runge in $Y \times X$, then the complex space $Y \times X \cup X \times Y$ is a homologically trivial space. The proof of this statement can be obtained using the cohomology Mayer-Vietoris sequence for the sets $X \times Y$, $Y \times X$ and their union and intersection and the Runge property of $Y \times Y$ in $Y \times X$.

Indead, using the same arguments as in the example in [3], we can construct the cohomology Mayer-Vietoris sequence

$$\begin{aligned} \ldots \to H^{k-1}(Y \times Y;\mathcal{F}) \to H^k(Y \times X \cup X \times Y;\mathcal{F}) \to \\ \to H^k(Y \times X;\mathcal{F}) \oplus H^k(X \times Y;\mathcal{F}) \to H^k(Y \times Y;\mathcal{F}) \to \ldots \end{aligned}$$

for each coherent sheaf $\mathcal{F}$ on $Y \times X \cup X \times Y$ and then from Theorem B of Cartan it follows that $H^k(Y \times X \cup X \times Y;\mathcal{F}) = 0$ for $k \geq 2$.

As $Y \times Y$ is relatively Runge in $Y \times X$, the restriction

$$H^0(Y \times X; \mathcal{F}) \to H^0(Y \times Y; \mathcal{F})$$

will admit a dense image. As the homomorphism

$$H^0(Y \times Y; \mathcal{F}) \to H^1(Y \times X \cup X \times Y; \mathcal{F})$$

in the sequence of Mayer-Vietoris is a surjection and a continuous map it follows that the topological vector space $H^1(Y \times X \cup X \times Y; \mathcal{F})$ has a trivial topology. Then from Proposition B there follows that the complex space $Y \times X \cup X \times Y$ is homologically trivial space. Note, that as is noted in [2] and [3] in such an way can be obtained not Stein spaces. An example is the case when X is the complex plane $\mathbb{C}$ and Y is the open unit disc in $\mathbb{C}$.

Acknowledgments to Prof. S. Dimiev for his interest and support.

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Received 23.03.1992