Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 7, 1993, Fasc. 2

A Measurable Feedback Decomposition of Differential Inclusions

R. P. Ivanov, N. M. Kitanov

Presented by P. Kenderov

This work is closely related to [6]. We prove decomposition theorem for the solutions of the differential inclusion

$$\dot{x} \in F(t,x),\ x(t_o) = x_o$$

using the solutions of the parameterized system of differential equations

$$\dot{x} = u(t,x),\ x(t_o) = x_o,$$

where $u(t,x) \in F(t,x)$ is a measurable selection or measurable feedback control. This paper is related to the investigations of I. Ekeland and M. Valadier [2], A. La Donne and M. V. Marchi [7], A. A. Tolstonogov [9], A. Ornellas [8], where the decomposition of the differential inclusions uses functions which depend on (t,x,u) (u is an additional variable).

Statement of the Problem

Let $[t_o,T] \times D \subset \mathbf{R} \times \mathbf{R}^n$ be a closed domain , where $\mathbf{R}$ is the real line and $\mathbf{R}^n$ is the Euclidean space. For $(t,x) \in [t_o,T] \times D$ we are going to consider the following differential inclusion:

$$\dot{x} \in F(t,x),\ x(t_o) = x_o, \tag{1}$$

where $x \in \mathbf{R}^n$, $(t,x) \in [t_o,T] \times \mathbf{R}^n$. Suppose $F(t,x)$ is a jointly measurable multi-function in (t,x) with compact values.

A solution of the differential inclusion (1) is said to be any absolutely continuous function $x(t)$ which for almost all t satisfies the differential inclusion (1), i.e.

$$\dot{x} \in F(t,x(t)) \text{ and } x(t_o) = x_o.$$

Definition 1. ([5]) The point z belongs to the essential limit of the function $u(y)$ when $y \to x$ (denote $z \in \operatorname*{ess\,lim}_{x \to y} u(y)$, if for any set $N \subset \mathbf{R}^n$ with Lebesgue's measure equal to zero there exists a sequence $y_i \notin N$, i = 1,2, ..., for which

$$\lim_{k\to\infty} y_k = x \text{ and } \lim_{k\to\infty} u(y_k) = z.$$

Let us denote

$$U(t,x) = \{z \in \mathbf{R^n} | z = \operatorname*{ess\,lim}_{y \to x} u(t,y)\}, \tag{2}$$

for any single-valued and measurable function $u(t,x)$.

The multifunction $U(t,x)$ is a jointly measurable function in (t,x). It is also upper semicontinuous with respect to x and the supperposition $U(t,x(t))$ is a measurable function for every continuous function $x(t)$ (see [5]).

For every measurable selection $u(t,x) \in F(t,x)$ (f.e. see [4] for its existence) we consider the differential equation

$$\dot{x} = u(t,x),\ x(t_o) = x_o$$

which is equivalent to the following differential inclusion:

$$\dot{x} \in co\, U(t,x),\ x(t_o) = x_o, \tag{3}$$

where $U(t,x)$ is defined by (2) (see [4],[5]).

We denote by H and G the set of solutions of the differential inclusions (1) and (3) respectively. We are going to consider the problem when the sets H and G coincide. This is the problem of the feedback control decomposition of the solutions of the differential inclusion (1). We notice that the family of functions of two variables (t,x) is as narrow as the family of functions which depend on three variables (t,x,u).

Obviously, we have $H \supset G$ if $F(t,x)$ is an upper semicontinuous function in x with compact values.

Treatments of the decomposition problem for the differential inclusions different from the above mentioned approach are presented f.e. in [8], [9].

Definition 2. The upper semicontinuous multifunction $F(x)$ is said to be a Kuratowski function if

$$F(x) = \text{ess}\ \operatorname*{Lim\,sup}_{y \to x} F(y),$$

where Lim sup is a Kuratowski upper limit (for more details see [1]) and **ess** is an essential limit (see definition 1).

Main Result

We are going to show that under some specific conditions imposed on the multifunction $F(t,x)$ (it may be nonconvex) the solutions of the differential inclusion (1) indeed decompose.

Theorem. *Let $F(t,x)$ be a jointly measurable and a Kuratowski multifunction in x with compact values. Then, for every solution $y(t)$ of the differential inclusion (1), there exists a measurable selection $u(t,x) \in F(t,x)$ for which*

$$\dot{y}(t) \in U(t, y(t)),\ y(t_o) = x_o,$$

where

$$U(t,x) = \{z \in \mathbf{R}^n | z = \operatorname*{ess\,lim}_{x \to y} u(t,y)\}.$$

Proof. Let $y(t)$ be the solution of the differential inclusion (1), $\dot{y}(t) \in F(t, y(t))$ and $y(t_o) = x_o$.

It is well known, that the conditions of the theorem don't guarantee the existence of solution of the inclusion (1).

We construct a measurable selection $u(t,x) \in F(t,x)$ as:

$$\| u(t,x) - \dot{y}(t) \| = \min_{u \in F(t,x)} \| u - \dot{y}(t) \| .$$

Now, we have

$$u(t,x) \in \mathrm{Pr}_{F(t,x)} \dot{y}(t) = \operatorname*{Arg\,min}_{u \in F(t,x)} \| u - \dot{y}(t) \| .$$

The functions $F(t,x)$ and $\dot{y}(t)$ are measurable, hence we can apply the Luzin theorem (see f.e. [1], [4]). Thus:

1. There exist compact subsets $D_\epsilon \subseteq D$ and $T_\epsilon \subseteq [t_o, T]$ for which the Lebesgue's measures $\mu(D_\epsilon)$ and $\mu(T_\epsilon)$ satisfy the following inequalities:

$$\mu(D_\epsilon) \geq \mu(D) - \epsilon,\ \mu(T_\epsilon) \geq T - t_o - \epsilon,$$

where ϵ is an arbitrarily chosen positive number.

2. The restriction of the multifunction $F(t,x)$ on the set $T_\epsilon \times D_\epsilon$ is a continuous multifunction and the restriction of the function $\dot{y}(t)$ on the set T_ϵ is a continuous function.

It is easy to check that

$$\operatorname*{Arg\,min}_{u \in F(t,x)} \| u - \dot{y}(t) \|$$

is an upper semicontinuous multifunction with closed values if the function $F(t,x)$ and $\dot{y}(t)$ are continuous. Hence

$$\operatorname*{Arg\,min}_{u \in F(t,x)} \| u - \dot{y}(t) \|$$

is an upper semicontinuous multifunction on the set $T_\epsilon \times D_\epsilon$ with closed values. Recaling that the positive number ϵ was chosen arbitrarily, the multifunction

$\operatorname{Arg\,min}_{u\in F(t,x)} \| u - \dot{y}(t) \|$ is a jointly in (t, x) measurable function and there exists a measurable selection

$$u(t, x) \in \operatorname*{Arg\,min}_{u\in F(t,x)} \| u - \dot{y}(t) \|$$

For more details one can see [1], [3] and [4].

We have to show that $y(t)$ satisfies the following differential inclusion

$$\dot{y}(t) \in co\, U(t, y(t)),\ y(t_o) = x_o,$$

where $U(t, x)$ is defined by (2). For a.e. $t \in [t_o, T]$ we have that $\dot{y}(t)$ belongs to $F(t, y(t))$. As well as $F(t, x)$ is a Kuratowski function w.r. to x, for every set N with Lebesgue measure $\mu(N) = 0$ there exists a sequence $x_k(t) \notin N$, k $= 1,2,\ldots$, $\lim_{k\to\infty} x_k(t) = y(t)$ and its respective sequence $y_k(t) \in F(t, x_k(t)), k = 1, 2, \ldots$ for which

$$\lim_{k\to\infty} y_k(t) = \dot{y}(t) \in F(t, y(t)).$$

Under the following inclusion

$$u(t, x_k(t)) \in \operatorname*{Arg\,min}_{u\in F(t,x_k(t))} \| u - \dot{y}(t) \|$$

we obtain

$$\lim_{k\to\infty} \| u(t, x_k(t)) - \dot{y}(t) \| \leq \lim_{k\to\infty} \| y_k(t) - \dot{y}(t) \| = 0.$$

Q.E.D.

N o t e. The requirement $F(t, x)$ to be a Kuratowski function w.r. to x is important as it shows the following example:

Let us define the following multifunction:

$$G(t, x) = \begin{cases} \{-1; 1\}, & \text{if } x = t, \\ -1, & \text{if } x \geq 0, \\ 1, & \text{if } x \leq 0. \end{cases}$$

This function is not a Kuratowski function because the value $+1 \in G(y, y)$ cannot be obtained as a Kuratowski essential upper limit. The differential inclusion

$$\dot{x} \in G(t, x),\ x(0) = 0$$

has an unique solution $x(t) = t$ which cannot be obtained as a solution of the differential inclusion (3). It easy to check that

$$U(t, x) = \begin{cases} -1, & \text{if } x \geq 0, \\ +1, & \text{if } x \leq 0, \end{cases}$$

and the differential inclusion $\dot{x} \in G(t, x)$, $x(0) = 0$, has no solutions.

R e m a r k. The relaxed differential inclusion

$$\dot{x} \in co\, U(t, X),\ x(t_o) = x_o$$

always has a solution [4]. Under the proved theorem we should offer the following definition for the solutions of differential inclusions with a right-hand side as a Kuratowski function:

Definition 3. Let $F(t,x)$ be a Kuratowski function. A solution of the following differential inclusion

$$\dot{x} \in F(t,x),\ x(t_o) = x_o$$

is said to be any absolutely continuous function $x(t)$ which satisfies almost everywhere the differential inclusion

$$\dot{x} \in co\, U(t,x),\ x(t_o) = x_o,$$

where

$$U(t,x) = \{z \in \mathbf{R}^n | z = \operatorname*{ess\,lim}_{y \to x} u(t,y)\},$$

and $u(t,x) \in F(t,x)$ are all measurable selections.

References

[1] J.-P. Aubin, H. Frankowska. Set-Valued Analysis, Birkhäuser, Boston o Basel o Berlin, 1990.

[2] I. Ekeland, M. Valadier. Representation of set-valued mappings. *J. Math. Anal. Appl.*, **35**, 1971, 621–629.

[3] A.F. Filippov. Some Problems in the Theory of Optimal Regularities. *Vest. MGU*, **2**, 1959, 25–32 (in Russian).

[4] A.F. Filippov. Differential equations with discontinues right hand side, Moscow, Nauka, 1985, (in Russian).

[5] R.P. Ivanov. Measurable Strategies in Differential Games, *USSR Math. Sbornik*, **180**, **1**, 1989, 119–135 (in Russian).

[6] R.P. Ivanov, N.M. Kitanov. On the Decomposition of the Solution of Some Differential Inclusions with Convex Right-hand Side. *Proceedings of XXI spring Conference of the Union of Bulgarian Mathematicians*, Sofia (April 3-6 1992), Sofia, 1992, 66–73.

[7] A. Le Donne, M.V. Marchi. Representation of Lipschizian compact convex valued mappings. *Rend. Acc. Naz. Linzei*, **68**, 1980, 278–280.

[8] A. Ornelas. Nonconvex Problems for Differential Inclusions. *PhD* Thesis, SISA, Trieste, 1988.

[9] A.A. Tolstonogov. Differential Inclusions in Banach Space, Novosibirsk, Nauka, 1985, (in Russian).

Institute of Mathematics
Bulgarian Academy of Sciences
P.O.Box 373
1090 Sofia
Bulgaria

Received 26.04.1992