Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol. 7, 1993, Fasc. 2

Group Theoretic Study of Certain Generating Functions Involving Modified Laguerre Polynomials

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In this paper we derive some new generating functions of modified Laguerre polynomials as defined by Goyal by the application of L. Weisner's group-theoretic method with the suitable interpretatioin of the parameter m.

1. Introduction

The modified Laguerre polynomials [1] defined by

$$L_{a,b,m,n}(x) = \frac{b^n(m)_n}{n!}\ {}_1F_1(-n;m;ax/b) \tag{1}$$

satisfies the following ordinary differential equation [2]:

$$xD_x^2u + (m - ax/b)D_xu + \frac{a}{b}nu = 0. \tag{2}$$

The aim of the presented paper is to derive certain generating functions of the said polynomials by L. Weiner's [3] group theoretic method (which does not seem to have appeared in the earlier works). For previous works on the polynomials under consideration we may mention the works [3–5].

2. Group-theoretic method

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, m by $y\frac{\partial}{\partial y}$ and u by $v(x,y)$ in (2) we get the partial differential equation:

$$x\frac{\partial^2 v}{\partial x^2} + y\frac{\partial^2 v}{\partial x \partial y} - \frac{a}{b}x\frac{\partial v}{\partial x} + \frac{a}{b}nv = 0. \tag{3}$$

Thus $v(x,y) = L_{a,b,m,n}(x)y^m$ is a solution of (3) since $L_{a,b,m,n}(x)$ is a solution of (2). We now define the following linear partial differential operators:

$$A_1 = y\frac{\partial}{\partial y}$$

$$A_2 = \frac{b}{a}y\frac{\partial}{\partial x} - y \tag{4}$$

$$A_3 = xy^{-1}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - y^{-1}$$

such that

$$A_1[L_{a,b,m,n}(x)y^m] = mL_{a,b,m,n}(x)y^m$$

$$A_2[L_{a,b,m,n}(x)y^m] = -y^{m+1}L_{a,b,m+1,n}(x) \tag{5}$$

$$A_3[L_{a,b,m,n}(x)y^m] = (n+m+1)y^{m-1}L_{a,b,m-1,n}(x)$$

The commutator relations satisfied by A_i $(i = 1,2,3)$ are

$$[A_1, A_2] = A_2, \quad [A_1, A_3] = -A_3, \quad [A_2, A_3] = 1. \tag{6}$$

So from the above commutator relations we arrive at the following theorem:

Theorem. *The set of operators* $\{1, A_i(i = 1,2,3)\}$, *where* 1 *stands for the identity operator, generates a Lie-algebra* $\mathcal{L}$.

It can be easily shown that the partial differential operator:

$$L = x\frac{\partial^2}{\partial x^2} + y\frac{\partial^2}{\partial x \partial y} - \frac{a}{b}x\frac{\partial}{\partial x} + \frac{a}{b}n$$

which can be expressed as follows:

$$\frac{b}{a}L = A_2A_3 + A_1 - n - 1 \tag{7}$$

commutes with each A_i $(i = 1,2,3)$, i.e.

$$[baL, A_i] = 0. \tag{8}$$

The extended form of the groups generated by A_i $(i = 1,2,3)$ are as follows:

$$e^{a_1A_1} f(x,y) = f(x, e^{a_1}y)$$

(9)
$$e^{a_2A_2} f(x,y) = e^{-a_2y} f(x + \frac{b}{a}a_2y, y)$$

$$e^{a_3A_3} f(x,y) = (1 + \frac{a_3}{y})^{-1} f(x(1 + \frac{a_3}{y}), y + a_3).$$

Thus we get

(10)
$$e^{a_3A_3}e^{a_2A_2}e^{a_1A_1} f(x,y) =$$
$$= (1 + \frac{a_3}{y})^{-1} \; e^{-a_2y(1+\frac{a_3}{y})} \; f((1 + \frac{a_3}{y})(x + \frac{b}{a}a_2y),\; e^{a_1}y(1 + \frac{a_3}{y})).$$

3. Generating function

From (3) we see that $v(x,y) = L_{a,b,m,n}(x)y^m$ is a solution of the system:

(11)
$$Lv = 0$$
$$\text{and } (A_1 - m)v = 0.$$

From (8) one can easily verify that

$$S(\frac{b}{a}L(L_{a,b,m,n}(x)y^m)) = \frac{b}{a}L(S(L_{a,b,m,n}(x)y^m)) = 0,$$

where

$$S = e^{a_3A_3}\, e^{a_2A_2}\, e^{a_1A_1}.$$

Thus the transformation $S(L_{a,b,m,n}(x)y^m)$ is annihilated by $\frac{b}{a}L$.

Putting $a_1 = 0$ and writing $f(x,y) = L_{a,b,m,n}(x)y^m$ in (10) we get

(12)
$$e^{a_3A_3}e^{a_2A_2}(L_{a,b,m,n}(x)y^m)$$
$$= (1 + \frac{a_3}{y})^{-1} \exp(-a_2y(1 + \frac{a_3}{y}))\, L_{a,b,m,n}\{(1 + \frac{a_3}{y})(x + \frac{b}{a}a_2y)\}\; \{y(1 + \frac{a_3}{y})\}^m$$

But

$$e^{a_3A_3}e^{a_2A_2}(L_{a,b,m,n}(x)y^m)$$
$$= y^m \sum_{p=0}^{\infty} \frac{(-\frac{a_3}{y})^p}{p!}(1 - m - n - k)_p \sum_{k=0}^{\infty} \frac{(-a_2y)^k}{k!} L_{a,b,m+k-p,n}(x).$$

Equating (13) and (14) we get

$$\exp(-a_2y(1+\frac{a_3}{y}))\ L_{a,b,m,n}\{(1+\frac{a_3}{y})(x+\frac{b}{a}a_2y)\}\ (1+\frac{a_3}{y})^{m-1}$$
$$=\sum_{p=0}^{\infty}\frac{(-\frac{a_3}{y})^p}{p!}(1-m-n-k)_p\sum_{k=0}^{\infty}\frac{(-a_2y)^k}{k!}L_{a,b,m+k-p,n}(x).$$

Case I: Putting $a_3=0$ and then replacing $-a_2y$ by t in (14) we get

$$e^t\,L_{a,b,m,n}(x-\frac{b}{a}t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}L_{a,b,m+k,n}(x). \tag{13}$$

Case II: Putting $a_2=0$ and then replacing $-a_3/y$ by t in (15) we get

$$(1-t)^{m-1}\,L_{a,b,m,n}\{x(1-t)\}\ =\ \sum_{k=0}^{\infty}\frac{t^k}{k!}(1-m-n)_k\ L_{a,b,m-k,n}(x).$$

Case III: Taking $a_2a_3\neq 0$, without any loss of generality we can choose $-a_2y=t_1$ and $-a_3/y=t_2$ in (14) and we get

$$\begin{aligned}&e^{t_1(1-t_2)}\ L_{a,b,m,n}\{(1-t_2)(x-\frac{b}{a}t_1\}\ (1-t_2)^{m-1}\\ &=\sum_{p=0}^{\infty}\frac{(t_2)^p}{p!}(1-m-n-k)_p\ \sum_{k=0}^{\infty}\frac{(t_1)^k}{k!}\,L_{a,b,m+k-p,n}(x).\end{aligned} \tag{14}$$

4. Variants of the result (15)

It is evident from the commutator relation $[A_2,A_3]=1$ that the operators A_2,A_3 are non-commutative and as such

$$e^{a_2A_3}\,e^{a_3A_3}\ \neq\ e^{a_2A_2+a_3A_3}.$$

So the relation (14) will change, if their orders be interchanged which is done in this section. In fact, by interchanging the order of operators we get

$$\begin{aligned}&e^{a_2A_2}\,e^{a_3A_3}[L_{a,b,m,n}(x)y^m]\\ &=e^{-a_2y}\ L_{a,b,m,n}\{x+\frac{b}{a}a_2y)(1+\frac{a_3}{y})\}\ y^m\ (1+\frac{a_3}{y})^{m-1}\end{aligned} \tag{15}$$

But

$$\begin{aligned}&e^{a_2A_2}\,e^{a_3A_3}[L_{a,b,m,n}(x)y^m]\\ &=y^m\sum_{p=0}^{\infty}\frac{(-\frac{a_3}{y})^p}{p!}(1-m-n)_p\sum_{k=0}^{\infty}\frac{(-a_2y)^k}{k!}L_{a,b,m+k-p,n}(x).\end{aligned} \tag{16}$$

Thus by equating (16) and (17) we get

$$
\begin{aligned}
& e^{-a_2 y} L_{a,b,m,n}\{x + \frac{b}{a}a_2 y)(1 + \frac{a_3}{y})\} (1 + \frac{a_3}{y})^{m-1} \\
(17) \quad & = \sum_{p=0}^{\infty} \frac{(-\frac{a_3}{y})^p}{p!}(1 - m - n)_p \sum_{k=0}^{\infty} \frac{(-a_2 y)^k}{k!} L_{a,b,m+k-p,n}(x).
\end{aligned}
$$

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Received 05.05.1992

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