Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica
Balkanica

New series Vol.7, 1993, Fasc.3-4

On a Type of P-Sasakian Manifold

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Introduction

Let (M, g) be an n-dimensional Riemannian manifold admitting a 1–form η which satisfies the conditions

$$(\nabla_X \eta)(Y) - (\nabla_Y \eta)(x) = 0 \tag{1}$$

$$(\nabla_X \nabla_Y \eta)(Z) = - g(X, Z)\eta(Y) - g(X, Y)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z) \tag{2}$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. If moreover (M, g) admits a vector field ξ and a (1–1) tensor field ϕ such that

$$g(X, \xi) = \eta(X) \tag{3}$$

$$\eta(\xi) = 1 \tag{4}$$

$$\nabla_X \xi = \phi X \tag{5}$$

then such manifold is called a para Sasakian manifold or briefly a P-Sasakian manifold by I. Sato and K. Matsumoto [1], [2]. This paper deals with a type of P-Sasakian manifold in which

$$R(X, Y).R = 0 \tag{6}$$

where R is the curvature tensor and $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of manifold for tangent vectors X, Y. In this connection we mention the works of T. Adati and T. Miyazawa [3], [4], T. Adati and K. Matsumoto [5] who studied P-Sasakian manifolds. Also we can mention the works of K. Sekigawa [6], Z.I. Szabo [7], L. Verstraelen [8], M. Petrovic-Torgasev and L. Verstraelen [9] who studied Riemannian manifolds and hypersurfaces of such manifolds satisfying the condition $R(X,Y).R = 0$ or conditions similar to it. Sasakian manifolds satisfying $R(X,Y).R = 0$ have been studied by T. Takahashi [10].

Let (M,g) be an n-dimensional Riemannian manifold admitting a 1–form η which satisfies the condition

$$(\nabla_X \eta)(Y) = -\ g(X,Y) + \eta(X)\eta(Y).$$

If moreover (M,g) admits a vector field ξ and a (1–1) tensor field ϕ such that the conditions (3), (4) and (5) are satisfied, then it has been called a special P-Sasakian manifold [1], [2]. In this paper, it is proved that in a P-Sasakian manifold (M,g) if the relation (6) holds then the manifold is SP-Sasakian and also a symmetric P-Sasakian manifold M is an SP-Sasakian manifold, where SP-Sasakian manifold is the short form of the special P-Sasakian manifold.

1. Preliminaries

It is known [1],[2] that in a P-Sasakian manifold the following relations hold:

(1.1) $$\phi\xi = 0$$

(1.2) $$\phi^2 X = X - \eta(x)\xi$$

(1.3) $$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

(1.4) $$S(X,\xi) = -\ (n-1)\eta(X)$$

(1.5) $$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X)$$

(1.6) $$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi$$

(1.7) $$R(\xi,X)\xi = X - \eta(X)\xi$$

and

(1.8) $$R(X,Y)\xi = \eta(X)Y - \eta(Y)X$$

The above results will be used in the next section.

2. P-Sasakian manifold satisfying $R(X,Y).R = 0$

Suppose $R(X,Y).R = 0$. Now,

$$(2.1)\qquad \begin{aligned}(R(X,Y).R)(U,V)W &= R(X,Y)R(U,V)W - R(R(X,Y)U,V)W\\ &\quad - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W\end{aligned}$$

In virtue of (2.1) and (6) we get
(2.2)
$$R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W = 0$$

Therefore,

$$(2.3)\qquad \begin{aligned}&g[R(\xi,Y)R(U,V)W,\xi] - g[R(R(\xi,Y)U,V)W,\xi]\\ &-g[R(U,R(\xi,Y)V)W,\xi] - g[R(U,V)R(\xi,Y)W,\xi] = 0\end{aligned}$$

From (2.3) it follows that

$$(2.4)\qquad \begin{aligned}&'R(U,V,W,Y) - \eta(Y)\eta(R(U,V)W) + \eta(U)\eta(R(Y,V)W)\\ &+ \eta(V)\eta(R(U,Y)W) + \eta(W)\eta(R(U,V)Y) - g(Y,U)\eta(R(\xi,V)W)\\ &- g(Y,V)\eta(R(U,\xi)W) - g(Y,W)\eta(R(U,V)\xi) = 0,\end{aligned}$$

where $'R(U,V,W,Y,) = g(R(U,V)W,Y)$.

Putting $Y = U$ in (2.4) we get

$$(2.5)\qquad \begin{aligned}&'R(U,V,W,Y) - \eta(U)\eta(R(U,V)W) + \eta(U)\eta(R(U,V)W)\\ &+ \eta(V)\eta(R(U,U)W) + \eta(W)\eta(R(U,V)U) - g(U,U)\eta(R(\xi,V)W)\\ &- g(U,V)\eta(R(U,\xi)W) - g(U,W)\eta(R(U,V)\xi) = 0.\end{aligned}$$

Let $\{e_i\}$, $i = 1,2,\ldots,n$ be an orthonormal basis of the tangent space at any point. Then the sum for $1 \le i \le n$ of the relation (2.5) for $U = e_i$ gives

$$(2.6)\qquad \eta(R(\xi,V)W) = \eta(V)\eta(W) + \frac{1}{n-1}S(V,W).$$

Now from (1.5), (3) and (4) we have

$$(2.7)\qquad \eta(R(\xi,V)W) = \eta(V)\eta(W) - g(V,W).$$

From (2.6) and (2.7) we get

$$(2.8)\qquad S(V,W) = -(n-1)g(V,W)$$

Thus from (2.8) it follows that

$$r = -\ n(n-1).$$

Using (1.5) and (3) it follows from (2.4) that

$$\begin{aligned}&'R(U,V,W,Y) - \eta(Y)\eta(R(U,V)W) + \eta(U)\eta(R(Y,V)W)\\ &+ \eta(V)\eta(R(U,Y)W) + \eta(W)\eta(R(U,V)Y) - g(Y,U)\eta(R(\xi,V)W)\\ &- g(Y,V)\eta(R(U,\xi)W) = 0.\end{aligned} \tag{2.9}$$

Using (1.5) and (2.8) it follows from (2.9) that

$$'R(U,V,W,Y) = -[g(Y,U)g(V,W) - g(Y,V)g(U,W)]$$

Thus the manifold is of constant curvature. But we know that [4] if a P-Sasakian manifold is of constant curvature, than the manifold is an SP-Sasakian manifold. Hence we can state the following theorems:

Theorem 1. *If in a P-Sasakian manifold M the relation $R(X,Y).R = 0$ holds, then it is an SP-Sasakian manifold.*

Theorem 2. *If in a P-Sasakian manifold M the relation $R(X,Y).R = 0$ holds, then the manifold is of scalar curvature $-n(n-1)$.*

For a symmetric Riemannian manifold due to Cartan we have $\nabla R = 0$. Hence for such a manifold $R(X,Y).R = 0$ holds. Thus we have following corollary of the above theorem 1:

Corollary . *A symmetric P-Sasakian manifold M is an SP-Sasakian manifold.*

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Received 05.06.1992