Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New series Vol.7, 1993, Fasc.3-4

Weak Fragmentability

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We introduce the notion of weak fragmentability which is identified with the weak-Radon Nikodym property and also with the Pettis property in weak*-compact, convex subsets of a dual Banach space. Also we prove that the characteristic properties of a convex, weakly-fragmented set K are that $\overline{convF}^{w^*} = \overline{convF}^{\| \|}$ for every weak*-compact subset F of K and that $L = \overline{convextL}^{\| \|}$ for every weak*-compact subset L of K. Finally is proved that a convex weakly fragmented set is affine homeomorphic to a weak*-compact subset of the dual space of a Banach space not containing l_1.

Introduction

The central theme of this paper in the notion of weak-fragmentability (Definition 1) which is inspired by the fragmentability defined in [8] and is identified with the scalar point of continuity property ([9]) in weak*-compact subsets of dual Banach spaces.

By K. Musial ([7]) and L. Janicka ([6]) it is proved that the dual Banach spaces with the weak-Radon-Nicodym (w-R.N.P.) property are characterized as spaces with predual not containing l_1. Also it is known ([11]) that a dual Banach space has the w-R.N.P. if and only if its weak*-compact subsets are weakly fragmented. On the other hand , every convex, weakly fragmented subset of a dual Banach space is affine homeomorphic to a weak*-compact subset of a dual space with the w-R.N.P. (Corollary 13).

Every convex, weak*-compact subset K of a dual Banach space X^* is weakly fragmented if and only if it has the w-R.N.P. or equivalently if it is a Pettis set (Theorem 9). Other characteristic properties of a convex weakly fragmented set K are that $\overline{convF}^{w^*} = \overline{convF}^{\|\ \|}$ for every weak*-compact subset F of K and that $L = \overline{convextL}^{\|\ \|}$ for every convex, weak*-compact subset L of K (Theorem 9 and Corollary 10). Using a version of the Superlemma ([1], [2]) we prove that K is weakly fragmented if and only if for every $x^{**} \in X^{**}$ the set of extreme points of K which are also points of continuity of $x^{**}/_K : (K, w^*) \to \mathbb{R}$ is a dense , G_δ set in $(\text{ext}K, w^*)$ (Proposition 4).

Finally, we prove that every bounded linear operator $T : X \to Y^*$ from a Banach space X into a dual space Y^* such that the weak*-closure of the image of the unit ball of X is a weakly fragmented subset of Y^* factors through a dual space with predual not containing l_1 (Corollary 14). This can also be proved by using a theorem of E. Saab and P. Saab ([11]), here we give a direct proof, which uses the notion of weak-fragmentability.

Notations

Let Z be a topological Hausdorff space and f a real valued function on Z. If $A \subseteq Z$, the oscillation of f on A is $O(f, A) = \sup\{|f(y) - f(x)| : x, y \in A\}$ and the oscillations of f at a point $x \in Z$ is $O(f, x) = \inf\{O(f, U) : U \subseteq Z$ is open and $x \in U\}$. Obviously, f is continuous at x if and only if $O(f, x) = 0$. We denote by $f/_A$ the restriction of f to A.

Let X be a Banach space. We denote by X^* and X^{**} the dual and the second dual of X respectively. The closed unit ball of X is denoted by B_X and its surface by S_X. If A is a subset of the dual space X^* then we denote by $\overline{A}^{\|\ \|}$ the norm closure of A, by $\overline{A}^{w}$ the weak-closure of A by $\overline{A}^{w^*}$ the weak*-closure of A and by conv$A$ the convex hull of A. The symbol (A, w^*) means that A is endowed with the relative w*-topology. If K is a convex subset of X then the set of extreme points of K is denoted by ext$K$.

If F is a bounded subset of a dual space X^* then a w*-slice (or w*-open slice) of F is a set of the form $S(F, x, a) = \{ f \in F : f(x) > M(x, F) - a \}$ where $x \in X$, $a > 0$ and $M(x, F) = \sup\{f(x); f \in F\}$. The corresponding w*-closed slice of S is the set $S_1 = S_1(F, x, a) = \{ f \in F : f(x) \geq M(x, F) - a \}$.

Definition 1. Let X be a Banach space and F a non-empty subset of the dual space X^*. We say that F is *weakly fragmented* if for each non-empty subset A of F, $\varepsilon > 0$ and $x_1^{**}, \ldots, x_n^{**} \in X^{**}$ there exists a non-empty relatively open subset U of (A, w^*) such that $O(x_i^{**}, U) < \varepsilon$ for every $i = 1, \ldots, n$.

It is easy to check that F is w-fragmented if and only if for every non-empty relatively closed subset H of (F, w^*) there exists a non-empty relatively open subset U of (H, w^*) such that $O(x^{**}, U) < \varepsilon$ for every $i = 1, \ldots, n$.

Proposition 2. *Let X be a Banach space and F a w^*-compact subset of the dual space X^*. The following statements are equivalent:*

(i) F is w-fragmented

(ii) For every non-empty, w^*-compact subset H of F, $\varepsilon > 0$ and $x^{**} \in X^{**}$ there exists a non-empty relatively open subset U of (H, w^*) such that $O(x^{**}, U) < \varepsilon$.

(iii) For every non-empty, w^*-compact subset H of F and $x^{**} \in X^{**}$ the restriction of x^{**} to (H, w^*) has a point of continuity. (scalar continuity property), ([9]).

(iv) For every w^*-compact subset H of F and $x^{**} \in X^{**}$ the set of points of continuity of the map $x^{**}/_H : (H, w^*) \to \mathbb{R}$ is a dense G_δ subset of (H, w^*).

Proof. The implications (i) $\Rightarrow$ (ii) (iv) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) are obvious.

(ii) $\Rightarrow$ (iv). Let $n \in \mathbb{N}$ and Z_n be the set of all $x \in H$ for which there exists a relatively open subset U of (H, w^*) with $x \in U$ and $O(x^{**}, U) < \frac{1}{n}$. Clearly the sets Z_n, $n \in \mathbb{N}$ are relatively open subsets of (H, w^*) and from (ii) they are dense in (H, w^*). The set $\cap_{n=1}^{\infty} Z_n$ is precisely the set of points of continuity of $x^{**}/_H : (H, w^*) \to \mathbb{R}$ and it is a dense, G_δ subset of (H, w^*), because (H, w^*) is a Baire space.

(iv) $\Rightarrow$ (i). For each $i = 1, \ldots, n$ let T_i be the set of points of continuity of $x_i^{**}/H : (H, w^*) \to \mathbb{R}$. From (iv) the sets T_i are dense G_δ in (H, w^*). Hence $\bigcap_{i=1}^{n} T_i$ is also dense, G_δ in (H, w^*). Let $e \in T$. Then there exists a relatively open subset U of (H, w^*) such that $e \in U$ and $O(x_i^{**}, U) < \varepsilon$ for every $i = 1, \ldots, n$.

We will prove that the set of the extreme points of K which are also points of continuity of $x^{**}/_K : (K, w^*) \to \mathbb{R}$ for some $x^{**} \in X^{**}$ is dense, G_δ in $(\text{ext}K, w^*)$ if K is a w*-compact, w-fragmented, convex subset of a dual space. This can be proved using analogous arguments as in the proof of proposition 8 in [11]. We will give a more elegant proof for this result using a version of the Superlemma [1], [2].

Lemma 3. *Let X be a Banach space, K, K_0 and K_1 are w^*-compact, convex subsets of X^*, $\varepsilon > 0$ and $x_1^{**}, \ldots, x_n^{**} \in S_{X^{**}}$. Suppose that:*

1. K_0 is a subset of K and $O(x_i^{**}, K_0) < \varepsilon$ for every $i = 1, \ldots, n$.

2. K is not a subset of K_1.

3. K is a subset of $\mathrm{conv}(K_0 \cup K_1)$*. Then there exists a w^*-slice S of K that intersects K_0 and $O(x_i^{**}, S) < \varepsilon$ for every $i = 1, \ldots, n$.*

The proof is virtually identical with that of the Superlemma (weak* version). (See [3] Theorem 3.4.1.(w*)).

Proposition 4. *Let X be a Banach space, K is a w^*-compact, w - fragmented, convex subset of X^*, and $x^{**} \in X^{**}$. Then the set $T \cap \mathrm{ext}K$ is dense, G_δ in $(\mathrm{ext}K, w^*)$, where T is the set of the points of continuity of $x^{**}/_K : (K, w^*) \to \mathbb{R}$. Consequently, $K = \overline{conv(T \cap extK)}^{w^*}$.*

Proof. Let $E = \mathrm{ext}K$. For each $\varepsilon > 0$ let B_ε be the set of elements of E which have a w*-open neighborhood V such that $O(x^{**}, V \cap K) < \varepsilon$. Every B_ε is open in (E, w^*) and we will prove that B_ε is also dense. Let W be a w*-open subset of X^* with $W \cap E \neq \emptyset$. Since K is w-fragmented there exists a w*-open subset U of X^* such that $U \subseteq W$, $U \cap E \neq \emptyset$ and $O(x^{**}, \overline{conv}^{w^*}(U \cap E)) < \varepsilon$, ([11] Proposition 7). Define $K_0 = \overline{conv(U \cap E)}^{w^*}$ and $K_1 = \overline{conv(E \backslash U)}^{w^*}$

The sets K_0, K_1, K satisfy the properties 1, 2, 3 of Lemma 3, hence we can find a w*-slice S of K that intersects K_0, misses K_1 and $O(x^{**}, S) < \varepsilon$. Let $u \in E \cap S$. Since $O(x^{**}, S) < \varepsilon$, we have that $U \in B_\varepsilon$. Also $u \in E \cap U \subseteq W$ since $K_1 \cap S = \emptyset$. Therefore B_ε is dense in (E, w^*).

Finally, $\mathrm{ext}K \cap T = \cap_n B_{1/n}$ is dense, G_δ in $(\mathrm{ext}K, w^*)$, since $(\mathrm{ext}K, w^*)$ is a Baire space.

Corollary 5. *Let A be a bounded subset of a dual space X^*. If $K = \overline{convA}^{w^*}$ is w-fragmented then for every $\varepsilon > 0$ and $x_1^{**}, \ldots, x_n^{**} \in X^{**}$ there exists a w^*-slice S of K such that $S \cap A \neq \emptyset$ and $O(x_i^{**}, S) < \varepsilon$ for every $i = 1, \ldots, n$.*

Proof. Let E_i $i = 1, \ldots, n$ be the sets of extreme points of K which are points of continuity of $x_i^{**}/_K : (K, w^*) \to \mathbb{R}$ respectively. By Proposition 4 the sets E_i are dense, G_δ in $(\mathrm{ext}K, w^*)$ for every $i = 1, \ldots, n$. Since $(\mathrm{ext}K, w^*)$ is a Baire space the set $E = \bigcap_{i=1}^{n} E_i$ is also dense, G_δ in $(\mathrm{ext}K, w^*)$. Let $e \in E$, then there exists a w*-slice S of K such that $e \in S$ and the corresponding closed slice S_1 of S has $O(x_i^{**}, S_1) < \varepsilon$ for every $i = 1, \ldots, n$.

Of course $S \cap A \neq \emptyset$ and $O(x_i^{**}, \overline{conv(S \cap A)}^{w^*}) < \varepsilon$ for every $i = 1, \ldots n$.

Proposition 6. *Let F be a w^*-compact subset of a dual Banach space*

X^*, such that $\overline{\mathrm{conv}F}^{w^*}$ is w-fragmented. Then $\overline{convF}^{w^*} = \overline{convF}^{\|\ \|}$.

P r o o f. Let $f \in K = \overline{convF}^{w^*}$. Then there is a regular Borel probability measure μ on (F, w^*) whose resultant $\tau(\mu)$ is equal to f. Let $\varepsilon > 0$ and $x_1^{**}, \ldots, x_n^{**} \in X^{**}$. The support M of μ is a nonempty, w*-compact subset of F and $M_1 = \overline{convM}^{w^*}$ is w-fragmented subset of K. From Corollary 5 there exists a w*-closed slice S_1 of M_1such that $O(x_i^{**}, \overline{conv}^{w^*}(S_1 \cap M)) < \varepsilon$ for every $i = 1, \ldots, n$ and $S_1 \cap M \neq \emptyset$. Therefore $\mu(S_1 \cap M) > 0$.

Let $\mathbf{A}$ be a maximal disjoint family of w*-compact subsets A of F such that $\mu(A) > 0$ and $O(x_i^{**}, \overline{conv}^{w^*} A) < \varepsilon$ for every $i = 1, \ldots, n$. Then $\mathbf{A}$ is countable and we will show that $\mu(\cup \mathbf{A}) = 1$. If $\mu(\cup \mathbf{A}) < 1$, then by the regularity of μ there exist a w*-compact subset L of F such that $L \subseteq F \setminus \cup \mathbf{A}$. and $\mu(L) > 0$. Let μ_1 the Borel measure on F with $\mu_1(B) = \mu(L \cap B)$ for every w*-Borel subset of F. If C is the support of μ_1, then C is a non-empty, w*-compact subset of L and $C_1 = \overline{conv}^{w^*} C$ is w-fragmented. From Corollary 5 there is a w*-closed slice S_1 of C_1 such that $O(x_i^{**}, \overline{conv}^{w^*}(S_1 \cap C)) < \varepsilon$ and $S_1 \cap C \neq \emptyset$. Hence $\mu(S_1 \cap C) > 0$. Therefore $\mathbf{A} \cup \{S_1 \cap C\}$ is properly larger than $\mathbf{A}$, contradicting maximality. Hence $\mu(\cup \mathbf{A}) = 1$.

Let $A_1, \ldots, A_k \in \mathbf{A}$ so that $\sum_{i=1}^{k} \mu(A_i) > 1 - \varepsilon$. Then $\mu = t_1\mu_1 + \cdots + t_k\mu_k + t_{k+1}\lambda$, where μ_i for $i = 1, \ldots, k$ and λ are regular, Borel, probability measures of (F, w^*) with $\mu_i(A_i) = 1$, $0 \leq t_i \leq 1$ for $i = 1, \ldots, k$, $t_{k+1} < \varepsilon$ and $\sum_{i=1}^{k} t_i = 1$. Therefore $f = \tau(\mu) = t_1\tau(\mu_1) + \cdots + t_k\tau(\mu_k) + t_{k+1}\tau(\lambda)$. Choose $f_i \in A_i$ for $i = 1, \ldots, k$ and $f_{k+1} \in F$. Since $\tau(\mu_i) \in \overline{convA_i}^{w^*}$, $i = 1, \ldots, k$ we have that $x_j(f_i^{**} - \tau(\mu_i)) < \varepsilon$ for every $j = 1, \ldots, n$ and $i = 1, \ldots, k$. Hence

$$x_j^{**}\left(f - \sum_{i=1}^{k+1} f_i\right) = \sum_{i=1}^{k} t_i x_j^{**}(\tau(\mu_i) - f_i) + t_{k+1} x_j^{**}(\tau(\lambda) - f_{k+1}) < \varepsilon + \operatorname{diam} F$$

for every $j = 1, \ldots, n$.
Since $\varepsilon > 0$ and $x_1^{**}, \ldots, x_n^{**}$ are arbitrary we conclude that $f \in \overline{convF}^{w^*} = \overline{convF}^{\|\ \|}$.

R e m a r k 7. For every w*-compact, w-fragmented, convex subset K of a dual Banach space we have that $K = \overline{convextK}^{\|\ \|}$. This can be proved with the same arguments as in the proof of Proposition 3.1 in [5]. Proposition 7 can be proved easily using the previous result. We gave a direct proof which has ideas of an analogous result of I. N a m i o k a [8].

In the following we will prove that the property which is described in Proposition 6 is a characteristic property of convex w-fragmented sets.

Definition 8. Let X be a Banach space and K a w* compact set of the dual space X^*.
K is called a Pettis set if the identity map $I : (K, w^*) \to X^*$ is universally scalarly measurable, i.e. for each $x^{**} \in X^{**}$ the function $x^{**}/_K : (K, w^*) \to \mathbb{R}$ is μ-measurable for every Radon probability measure μ on (K, w^*).
K is called weak- Radon- Nikodym set (w-R.N. set) if for every probability space (Ω, F, μ) and every bounded operator $T : L_1(\mu) \to X^*$ for which $T(x_E/\mu(E))$ belongs to K for each non-null measurable set E, the operator T is represented by a Pettis integrable function with values in K.

Theorem 9. *Let K be a w^*-compact convex subset of a dual space X^*. The following are equivalent:*
(i) K is w-fragmented.
(ii) K is Pettis set.
(iii) K has the weak Radon-Nikodim property.
(iv) For each w^*-compact subset F of K we have $\overline{convF}^{w^*} = \overline{convF}^{\| \|}$.

Proof. (i) $\Leftrightarrow$ (iii) K is w-fragmented if and only if K has the scalar point of continuity property, from Proposition 2. Hence we have the equivalence from the results in [9] and [10].

(ii) $\Leftrightarrow$ (iv) This equivalence is proved in [12].

(ii) $\Rightarrow$ (i) In [12] is proved that a w*-compact subset F of X^* is a Pettis set if and only if $\overline{conv}^{w^*}(K \cup (-K))$ is a Pettis set. Since every w*-compact absolutely convex subset of X^* is w-fragmented ([9]) we have that every Pettis set is w-fragmented.

(i) $\Rightarrow$ (iv) It is obvious from Proposition 6.

Corollary 10. *A convex, w^*-compact subset K of a dual Banach space X^* is w-fragmented if and only if $L = \overline{convextL}^{\| \|}$ for every convex subset L of K.*

The following Theorem is unfluenced by the methods of W.Davis- I.Figiel- W.B.Johnson- A.Pelczynski [4] and I.Namioka [8].

Lemma 11. *Let F be a w^*-compact, w-fragmented subset of a dual space X^*, B the unit ball of X^*, $\varepsilon > 0$, $\delta > 0$ and $x_1^{**}, \dots, x_n^{**} \in S_{X^{**}}$. If A is a* *non-empty subset of $K + \delta B^*$ then there is a w^*-open subset V of X^* such that $V \cap A \neq \emptyset$ and $O(x_i^{**}, V \cap A) < 2\delta + \varepsilon$ for every $i = 1, \ldots, n$.*

Proof. We can assume that A is w*-compact. We define the function $f : F \times \delta B \to X^*$ by $f(x^*, y^*) = x^* + y^*$. If $F \times \delta B$ has the product of the weak*-topologies and X^* the weak*-topology the function f is continuous and of course $A \subseteq f(F \times \delta B)$.

Let M be a minimal compact subset of $F \times \delta B$ such that $f(M) = A$. Then $\Pi_1(M)$ (where Π_1 is the projection map) is w-fragmented subset of F, hence there exists a relatively w*-open subset U of F such that $U \cap \Pi_1(M) \neq \emptyset$ and $O(x_i^{**}, V \cap \Pi_1(M)) < \varepsilon$ for every $i = 1, \ldots, n$. By the minimality of M, $f(M \backslash \Pi_1^{-1}(V))$ is a proper subset of A, hence $W = A \backslash f(M \backslash \Pi_1^{-1}(V)) \subseteq f(M \cap \Pi_1^{-1}(V))$ is a non-empty, relatively w*-open subset of A. Let x^*, $y^* \in W$, then $x^* = x_1^* + x_2^*$ and $y^* = y_1^* + y_2^*$ with $(x_1^*, x_2^*), (y_1^*, y_2^*) \in (V \cap \Pi_1^{-1}(M)) \times \delta B$. Then

$$x_i^{**}(x^* - y^*) = x_i^{**}(x_1^* - y_1^*) + x_i^{**}(x_2^* - y_2^*) < \varepsilon + 2\delta$$

and $O(x_i^{**}, W) < 2\delta + \varepsilon$ for every $i = 1, \ldots, n$ as was to be shown.

Theorem 12. *Let X be a Banach space and K a w^*-compact, w-fragmented, absolutely convex subset of a dual space X^*. Then there exists a bounded linear operator T from X onto a dense subspace of a Banach space E not containing l_1 such that $K \subseteq T^*(B)$, where B is the unit ball of E^*.*

Proof. Let V be the unit ball of X^*, $U_n = 2^n K + (1/2^n)V$ for every $n = 1, 2, \ldots$ and $\| \|_n$ a norm for X^* whose unit ball is U_n. The norms $\| \|_n$ are dual and equivalent to the original one on X^*. Let $p(x^*) = \sum_{n=1}^{\infty} \|x^*\|_n$ for every $x^* \in X^*$, $H = \{x^* \in X^* : p(x^*) < \infty\}$ and $U = \{x^* \in X^* : p(x^*) \leq 1\}$.

As proved in [8], (H, p) is a dual Banach space and U is a w*-compact, absolutely convex subset of X^* with $K \subseteq U$. We will prove that U is also w-fragmented. Let F be a non-empty, w*-compact subset of U, $\varepsilon > 0$ and $x^{**} \in X^{**}$. Fix $n \in \mathbb{N}$ such that $1/2^n < (\varepsilon/3)\|x^{**}\|$.
Then $F \subseteq U \subseteq U_n = 2^n K + (1/2^n)V \subseteq 2^n K + (\varepsilon/2\|x^{**}\|)V$. From Lemma 11 there is a relatively w*-open subset W of F with $O(x^{**}, W) < \varepsilon$. Hence U is a w-fragmented subset of X^*.

We will show that (H, p) has the w-R.N.P.. It is sufficient to prove that U is a w-fragmented subset of (H, p). Let F be a non-empty, w*-compact subset of U, $\varepsilon > 0$, and $x^{**} \in S_{H^*}$. If $m = \sup\{p(f) : f \in F\}$, then we choose an $f_0 \in F$

such that $p(f_0) > m - \varepsilon/3$ and consequently $n_0 \in \mathbb{N}$ such that $\sum_{n=1}^{n_0} \|f_0\|_n >$ $m - \varepsilon/3$. Write $q(g) = \sum_{n=1}^{n_0} \|g\|_n$ for every $g \in X^*$. Then q is an equivalent norm in E^* and also q is w*-lower semicontinuous. Let $W = \{f \in F : q(f) > m - \varepsilon/3\}$. Then $f_0 \in W$ and W is a relatively w*-open subset of F. We define $Z = (\sum_n X_n)_1$ where $X_n = (X^*, \| \|_n)$ and $\varphi : H \to Z$ by $\varphi(F) = (jf, jf, \ldots)$ where $j : H \to X^*$ is the inclusion map. Then $\varphi(H)$ is a closed linear subspace of Z and (H, p) is isometric to $\varphi(H)$. Hence $\varphi^* : (\sum_n X_n^*)_\infty \to H^*$ is onto H^* and therefore there exists $y^* = (x_1^*, x_2^*, \ldots) \in \sum_{n=1}^{\infty} X_n^*$ with $\|y^*\|_\infty \leq 1$ such that $y^*(f) = \sum_n x_n^*(f)$ for every $f \in H$. Let $z^* : X^* \to \mathbb{R}$ with $z^*(g) = \sum_{n=1}^{n_0} x_n^*(g)$ for every $g \in X^*$. Then $z^* \in X^{**}$, because the norm q is equivalent to the original norm of X^*. Since U is a w-fragmented subset of X^*, there exists a non-empty relatively w*-open subset G of W such that $O(z^*, G) < \varepsilon/3$. The set G is relatively w*-open in F and $O(x^*, G) < \varepsilon$. Indeed, let $g_1; g_2 \in G$, then

$$x^*(g_1 - g_2) = \sum_{n=1}^{\infty} x_n^*(g_1 - g_2) = \sum_{n=1}^{n_0} x_n^*(g_1 - g_2) + \sum_{n=n_0+1}^{\infty} x_n^*(g_1 - g_2)$$

$$< \varepsilon/3 + \sum_{n=n_0+1}^{\infty} \|g_1\|_n + \sum_{n=n_0+1}^{\infty} \|g_2\|_n < \varepsilon$$

because $g_1, g_2 \in W$. This shows that U is w-fragmented in (H, p).

Let $C(U)$ be the Banach space of all continuous scalar-valued functions on (U, w^*) with the supremum norm and let $R : X \to C(U)$ be the bounded linear operator defined by $R(x)(u) = u(x)$ for every $x \in X$ and $u \in U$. Let $E = \overline{R(X)}$ and $T : X \to E$ be the map which is obtained from R by restricting the range. Then $T^* : E^* \to X^*$ is an isometry onto (H, p) and $K \subseteq T^*(B)$ where B is the unit ball of E^*.

Corollary 13. *Every Pettis set (F, w^*) of a dual Banach space X^* is (affine) homeomorphic to a w^*-compact subset of the dual E^* of a Banach space E not containing $l1$.*

Proof. If F is a Pettis set then $K = \overline{conv(F \cap -F)}^{w^*}$ is also a Pettis set [1]. According to Theorems 9 and 12 there exist a space E not containing

l_1 and a bounded linear map $T : X \to E$ with dense range such that $K \subseteq T^*(B_{E^*})$. Since the restriction of $T^*/_{B_{E^*}} : (B_{E^*}, w^*) \to (T^*(B_{E^*}), w^*)$ is a homeomorphism we have that (F, w^*) is (affine) homeomorphic to a w*-compact subset of E^*.

Corollary 14. *Let X, Y be Banach spaces and $T : X \to Y^*$ a bounded linear operator such that $\overline{T(B_X)}^{w^*}$ is w-fragmented subset of Y^*. Then T factors through a dual Banach space E^* with E not containing l_1.*

Proof. Apply Theorem 12 to $K = \overline{T(B_X)}^{w^*}$. Then there exists a dual Banach space (H, p) with predual not containing l_1 such that $K \subseteq B_H$, $H \subseteq Y^*$ and the inclusion map $J : H \to Y^*$ be continuous. Hence $T = J \circ S$ where $S = J^{-1} \circ T : X^* \to H$.

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Received 17.06.1992