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On the Solvability of Random Volterra Inclusions

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Based on an abstract existence theorem for random operator inclusions, we give results on the solvability of nonlinear random Volterra inclusions (equations) of the general form

$$f(\omega)t \in u(\omega)t + \int_0^t k(\omega)(t-s)A(\omega)u(\omega)s ds, \; t \in [0,T], \omega \in \Omega$$

in a Hilbert space H, where A is a multi-(single-) valued pseudomonotone random operator, and Ω is a complete measure space.

0. Introduction

In this paper we study the problem of existence of random solutions of a nonlinear random Volterra inclusion. Deterministic problems of this type arise in many areas of applied mathematics.

More precisely we consider an inclusion of the form

$$f(\omega)t \in u(\omega)t + \int_0^t k(\omega)(t-s)A(\omega)u(\omega)s ds, \; t \in [0,T], \omega \in \Omega \tag{0.1}$$

in a Hilbert space H, where Ω is a complete measure space, A is a multivalued, pseudomonotone random operator, and k is a random kernel of positive type. The deterministic analogs have been studied by many authors, cf. V. Barbu [1], M.G. Crandall and J.A. Nohel [4] and more recently by N. Hirano [9], [10].

In Section 2, we prove a general existence theorem for an abstract random operator inclusion which generalizes recent results ([11], [15]) to the multivalued case. This plays a crucial role in the proof of the main theorem of the next section, where we study the problem of existence of random solutions for (0.1).

Finally, in Section 4, we prove existence and uniqueness of random solutions for the corresponding nonlinear Volterra equation.

1. Preliminaries

Let (Ω, A, μ) be a complete σ-finite measure space. Let X be a real separable Banach space, X^* its dual and (x, x^*) the pairing between $x^* \in X^*$ and $x \in X$.

Define the set

$$S_F^p = \{f \in L^p(\Omega; X) : f(\omega) \in F(\omega) \ a.e.\}$$

It can be shown that S_F^p is a closed subset of $L^p(\Omega; X)$, $1 \leq p < \infty$. The following Lemma is useful (for a proof see [7]).

Lemma 1.1. *Let $F : \Omega \to 2^X$ be a weakly measurable function, such that $F(\omega)$ is nonempty and closed for all $\omega \in \Omega$, and $1 \leq p < \infty$. If S_F^p is nonempty then there exists a sequence $\{f_n\}$ contained in S_F^p, such that $F(\omega) = cl\{f_n(\omega)\}$, for all $\omega \in \Omega$.*

Using S_F^1, we can define an integral for multifunctions:

$$\int_\Omega F(\omega)d\mu(\omega) = \{\int_\omega f(\omega)d\mu(\omega) : f \in S_F^1\},$$

where the vector integrals are taken in sense of Bochner.

Finally we denote by $B(\Omega, X)$ the set of all measurable mappings $\xi : \Omega \to X$ such that $\sup\{||\xi(\omega)|| : \omega \in \Omega\} < +\infty$.

As far as terminology is concerned, one can consult [3] or [8], for measurability and randomness notions, and [14] for monotonicity ones.

2. Multivalued perturbations of nonlinear monotone operators

For the definition of pseudomonotonicity for multivalued mappings and related results we refer to [14].

For the proof of the main result of this section we need the following.

Lemma 2.1. *Let Ω be a complete measure space. Let X be a reflexive, separable Banach space and D be a closed and convex subset of X with $0 \in D$. Let $L : \Omega \times D \to 2^{X^*}$ be a monotone, random operator such that L is l. s. c. (X^* taken with its weak topology), and $L(\omega)x$ is closed in X^* for each $\omega in \Omega$ and $x \in D$. Further suppose that for each $\omega \in \Omega$ there exists $v(\omega) \in L(\omega)0$ such that $\sup\{\|v(\omega)\| : \omega \in \Omega\} = m < \infty$.*
Let F be a finite dimensional subspace of X and $T : \Omega \times F \to 2^{X^*}$ be a pseudomonotone, l. s. c., bounded, coercive and jointly measurable operator. Then there exist $\xi \in B(\Omega, F)$ and $z \in T(\Omega)\xi(\omega)$ such that

$$(y + z, x - \xi(\omega) \geq 0 \text{ for each } \omega \in \Omega, x \in F \cap D,\ y \in L(\omega)x.$$

Proof. Define $G : \Omega \to 2^B$ by

$$G(\omega) = \{u \in B : (y + z, x - u) \geq 0), \text{ for each } x \in F \cap D,\ y \in L(\omega)x\},$$

where $B = \{u \in F : \|u\| \leq R\}$; it is not difficult to see that such an R exists.

To prove that $G(\omega)$ is measurable, it suffices to show the measurability of

$$\gamma(\omega) = \bigcap_{n \geq 0} \{u \in B : (v(\omega)x + \varphi_n(\omega)u, x - u) \geq 0\}$$

since by [6] (Thm 1) one has

$$T(\omega)u = cl\{\varphi_n(\omega)u\}$$

where $\varphi_n : \Omega \times (F \cap D) \to X^*$ are selectors of Caratheodory type of T.

Evidently, $\psi_n(\omega, u) := (v(\omega)x + \varphi_n(\omega)u,\ x - u)$ is measurable in ω and continuous in u, and hence by [8] (Thm. 6.4), G is a measurable. The result now follows from the Selection Theorem of K. Kuratowski and C. Ryll-Nardzewski [12]. ■

The following theorem is the stochastic analog of a result obtained by F. E. Browder ([2] Theorem 7.8). It also extends Theorem 1 of [11], to the case of a multivalued perturbation T.

Theorem 2.1. *Let Ω be complete, X a separable, reflexive, Banach space, D a closed, convex subset of X with $0 \in D$. Let $L : \Omega \times D \to 2^{X^*}$ be a* *maximal monotone random operator such that L is l. s. c. (X^* taken with its weak topology). Suppose that for each $\omega \in \Omega$ there exists $u(\omega) \in L(\omega)0$ such that $\sup\{\|u(\omega)\| : \omega \in \Omega = M\} < \infty$. Let $T : \Omega \times X \to 2^{X^*}$ be a pseudomonotone, l. s. c., bounded, coercive and jointly measurable operator. Then for each $\eta \in (\Omega, X^*)$ there exists $\xi \in B(\Omega, X)$ such that*

$$\eta(\omega) \in L(\omega)\xi(\omega) + T(\omega)\xi(\omega), \text{ for all } \omega \in \Omega.$$

Proof. We may assume that $\eta(\omega) = 0$ for all $\omega \in \Omega$. Let $\{F_k\}$ be an increasing sequence of finite dimensional subspaces of X such that $\cup_k F_k$ is dense in X and $\cup_k(D \cap F_k)$ is dense in D.
By Lemma 2.1, for each k, there exist $\xi_k \in B(\Omega, F_k)$ and $z_k \in T(\omega)\xi_k(\omega)$ such that

$$(v + z_k, y - \xi_k(\omega)) \geq 0, \tag{2.1}$$

for all $\omega \in \Omega$, $y \in F_k \cap D, v \in L(\omega)y$.

From this inequality we get that

$$\psi_k(\omega) = \{\xi_i(\omega) : i \geq k\}$$

are weakly measurable and hence the same holds for

$$\Psi(\omega) = \bigcap_k \psi_k(\omega).$$

Therefore Ψ admits a measurable selection ([5]p. 149).
For fixed $\omega \in \Omega$, there is a subsequence $\{\xi_m(\omega)\}$ of $\{\xi_k(\omega)\}$ such that $\xi_m(\omega) \rightharpoonup \xi(\omega)$.
Since T is bounded, there exists a subsequence $\{z_l\}$ of $\{z_m\} \subset \{T(\omega)\xi(\omega)\}$, such that

$$z_l \rightharpoonup z^*.$$

From (2.1) we get that

$$(z_l, \xi_l(\omega) - y) \leq (v, y - \xi_l(\omega)). \tag{2.2}$$

for all $y \in D \cap F_l$ and $v \in L(\omega)y$.

Using the l. s. c of L and setting $y = \xi(\omega)$, we get

$$\limsup\,(z_l, \xi_l(\omega) - \xi(\omega)) \leq 0. \tag{2.3}$$

From the pseudomonotonicity of T we conclude that for each $x \in D$ there exists a $z_x \in T(\omega\xi(\omega)$ such that

$$(z_x, \xi(\omega) - x) \leq \liminf\,(z_l, \xi_l(\omega) - x) \tag{2.4}$$

Taking $x = \xi(\omega)$, we easily conclude that

$$\lim\,(z_l, \xi_l(\omega) - x) \;=\; (z^*, \xi(\omega) - x), \tag{2.5}$$

and hence

$$(z_x, \xi(\omega) - x) \;=\; (z^*, \xi(\omega) - x). \tag{2.6}$$

From [2], Proposition 7.1, we get that for each $\omega \in \Omega$, there exists a $z(\omega) \in T(\omega)\xi(\omega)$, such that

$$(z(\omega), \xi(\omega) - x) \;=\; (z^*, \xi(\omega) - x) \text{ for all } x \in D, \tag{2.7}$$

and therefore, by (2.1) we have

$$(z^*, \xi(\omega) - y) \;=\; (v, y - \xi(\omega)). \tag{2.8}$$

(2.7) and (2.8) provide the existence of $z(\omega) \in T(\omega)\xi(\omega)$, so that

$$(v + z(\omega), y - \xi(\omega)) \;\geq\; 0 \text{ for all } y \in D,\; v \in L(\omega)y. \tag{2.9}$$

Since L is maximal monotone, it follows that

$$-z(\omega) \;\in\; L(\omega)\xi(\omega) \text{ where } z(\omega) \in T(\omega)\xi(\omega),$$

i.e. $0 \;\in\; L(\omega)\xi(\omega) \;+\; T(\omega)\xi(\omega)$. ■

3. Nonlinear random Volterra inclusions

In this section we consider the existence of random solutions to the nonlinear random Volterra integral inclusion

$$f(\omega)t \in u(\omega)t + \int_0^t k(\omega)(t-s)A(\omega)u(\omega)s\,ds,\; t \in [0,T], \omega \in \Omega, \tag{3.1}$$

in a separable Hilbert space H, where $A : \Omega \times H \to 2^H$ is a nonlinear random operator, with $D(A) = \Omega \times H$. $k : \Omega \times [0,T] \to R$ is a random kernel, and f is a random function from $\Omega \times [0,T]$ into H.

Recently, N. Hirano [9] obtained existence results for the deterministic analog of (3.1) in case A is a pseudomonotone, multivalued operator on H.

Throughout this section H will denote a real separable Hilbert space, with norm $|\cdot|$ and inner product $(\cdot,\cdot)$. Let $\|\cdot\|_2$ and $(\cdot,\cdot)_2$ be the norm and the inner product, respectively, of the space $L^2(\Omega, L^2(0,T;H))$.

For each $k \in L^\infty(\Omega, L^1(0,T))$

$$L_k : \Omega \times L^2(\Omega, L^2(0,T;H)) \to L^2(\Omega, L^2(0,T;H))$$

denotes the linear, continuous, random operator defined by

$$(L_k(\omega)f)(\omega,t) = \int_0^T k(\omega)(t-s)f(\omega,s)ds,$$

for all $\omega \in \Omega$, $t \in [0,T]$.

Remark. The notations $g(\omega)t$ and $g(\omega,t)$ both have the same meaning.

We next state the hypotheses for the kernel:

(K1) $k \in L^\infty(\Omega, L^1(0,T))$ is of positive type, i.e. for each $f \in L^2(\Omega, L^2(0,T;H))$ we have

$$\int_0^t f(\omega)\rho \int_0^\rho k(\omega)(t-s)f(\omega)s ds d\rho \geq 0, \text{ for each } \omega \in \Omega,\ t \in [0,T],$$

(K2) L_k^* is injective, i.e. $L_k^*(\omega)f = 0$, for all $\omega \in \Omega$, implies $f = 0$.

Let $\tilde{H} = L^2(\Omega, L^2(0,T;H))$. Denote by $\tilde{A} : \Omega \times \tilde{H} \to 2^{\tilde{H}}$ the operator defined by

$$\tilde{A}(\omega)u = \{w \in \tilde{H} : w(\omega)t \in A(\omega)u(\omega)t \text{ a.e. on } [0,T], \text{ for all } \omega \in \Omega\}$$

The hypotheses on A are

(A1) there exist c_1, $c_2 \in B(\Omega, R)$ such that

$$|y| \leq c_1(\omega) + c_2(\omega)|x| \text{ for all } y \in A(\omega)x,\ x \in H, \omega \in \Omega.$$

(A2) $\liminf\limits_{x\to\infty} \dfrac{(x,y)}{|x|} > -d$, for all $\omega \in \Omega$ where $d \in R^+$ and $y \in A(\omega)x$.

Finally the hypotheses of $\tilde{A}$ (corresponding to (A1) and (A1)) are:

($\tilde{\mathrm{A}}1$) there exist $\tilde{c}_1, \tilde{c}_2 \in B(\Omega, R)$ such that

$$\|v\|_2 \leq \tilde{c}_1(\omega) + \tilde{c}_2(\omega)\|u\|_2 \text{ for all } v \in \tilde{A}(\omega)u,\ u \in \tilde{H}, \omega \in \Omega.$$

($\tilde{\mathrm{A}}2$) $\liminf_{\|u\|_2 \to \infty} \dfrac{(v,u)}{\|u\|_2} > -\tilde{d}$, for all $\omega \in \Omega$ where $\tilde{d} \in R^+$ and $v \in \tilde{A}(\omega)u$.

Remark. It is obvious that if A satisfies (A1) and (A2) , then $\tilde{A}$ satisfies ($\tilde{\mathrm{A}}1$) and ($\tilde{\mathrm{A}}2$).

Now we are in a position to state and prove the following

Theorem 3.1. *Let $\tilde{A} : \Omega \times \tilde{H} \to 2^{\tilde{H}}$ be a pseudomonotone random operator with $D(\tilde{A}) = \Omega \times \tilde{H}$ that satisfies ($\tilde{A}1$) and ($\tilde{A}2$).*
Let $\tilde{A}(\omega, \cdot)$ be l. s. c. from $\tilde{H}$ into $\tilde{H}$ and u. s. c. from $\tilde{H}$ into $\tilde{H}_w$. Suppose that k is a continuous mapping satisfying (K1), and that there exists $\delta(\omega) \in (0,1)$ such that

$$\text{(K3)} \qquad \tilde{c}_2(\omega)\|k(\omega)\|_{L^1(0,T)} \leq \delta(\omega), \text{ for all } \omega \in \Omega$$

Then, for each $f \in \tilde{H}$ the inclusion (3.1), has a random solution.

Proof. The inclusion (3.1) can be written in operator form as

$$\text{(3.2)} \qquad f(\omega) \in u(\omega) + L_k(\omega)\tilde{A}(\omega)u(\omega), \text{ for all } \omega \in \Omega$$

By ([9] (Lemma 4)), the equation (3.2) has, for each $\omega \in \Omega$, a solution $u(\omega)$ on $[0,T]$ if and only if the equation

$$\text{(3.3)} \qquad 0 \in v(\omega) + \tilde{A}(\omega)\left(L_k(\omega)v(\omega), +f(\omega)\right), \text{ for all } \omega \in \Omega$$

has a solution $v(\omega)$ on $[0,T]$.

For the sake of simplicity, we write $L(\omega)$ instead of $L_k(\omega)$, and for each $n \geq 1$, we set $L_n(\omega) = (L(\omega) + \frac{1}{n})I$.

Since $L_n^*(\omega)$ is bijective, (3.3) is equivalent to

$$\text{(3.4)} \qquad 0 \in L_n^*(\omega)v(\omega) + L_n^*(\omega)\tilde{A}(\omega)\left[L_n(\omega)v(\omega) + f(\omega)\right] \text{ for all } \omega \in \Omega$$

From ([9] (Theorem 1)), we easily obtain that $L_n^*(\omega) - \frac{1}{n}I$ is a linear, maximal monotone operator, continuous in v, and measurable in ω, that also satisfies the remaining conditions on L of Theorem 2.1.

Furthermore $\frac{1}{n}I + L_n^*(\omega)\tilde{A}(\omega)[L_n(\omega)+f(\omega)]$ is a pseudomonotone, coercive, bounded operator and by a result of [13] the hypotheses on $\tilde{A}$ imply the joint-measurability of this operator.

Hence, by Theorem 2.1, for each $n \geq 1$, there exists a $\xi_n \in B(\Omega, \tilde{H})$ such that

$$(3.5) \qquad 0 \in L_n^*(\omega)\xi(\omega) + L_n^*(\omega)A(\tilde{\omega})\left[L_n(\omega)\xi(\omega) + f(\omega)\right] \text{ for all } \omega \in \Omega$$

Using (3.5), ($\tilde{A}1$) and (K3) we get that

$$\|\xi_n(\omega)\|_2 < M\left(\delta(\omega), \tilde{c}_1(\omega), \tilde{c}_2(\omega), \|f(\omega)\|_2\right) := M$$

where $M > 0$, for all $\omega \in \Omega$, $n \geq 1$.

We therefore conclude that there exist a subsequence $\{w_m(\omega)\}$ of $\{w_n(\omega)\}$ and a measurable mapping $w : \Omega \to B_n := \{x \in \tilde{H} : \|x\|_2 \leq M\}$, where $w_m(\omega) = -\xi_m(\omega)$, such that

$$w_m(\omega) \longrightarrow w(\omega)$$

From this relation we arrive at

$$\limsup\,(w_m(\omega)\ ,\ L_m(\omega)\xi_m(\omega) - L(\omega)u(\omega)) \leq 0, \text{ for all } \omega \in \Omega.$$

By [9] (Lemma 2) we obtain that

$$w(\omega) \in A(\tilde{\omega})\left[L(\omega)u(\omega) + f(\omega)\right] \text{ for all } \omega \in \Omega \text{ i.e.}$$

$$0 \in u(\omega) + A(\tilde{\omega})\left[L(\omega)u(\omega) + f(\omega)\right] \text{ for all } \omega \in \Omega \blacksquare$$

Remark 3.1. (i) The operator $L_n^*\tilde{A}L_n$ is l. s. c. (respectively, u. s. c.) provided that $\tilde{A}$ is l. s. c. (respectively, u. s. c.)
(ii) If $\tilde{A}$ is known to be jointly-measurable, then we can avoid the u. s. c. hypothesis on it.

Example 3.1. Consider the Initial Value Problem for the differential inclusion:

$$(*) \qquad \begin{cases} 0 \in \dfrac{du(\omega)t}{dt} + M(\omega,t)u(\omega)t + \displaystyle\int_0^T a(\omega)(t,s)C(\omega, u(\omega)s)ds \\ \qquad u(\omega)0 = u_0 \end{cases}$$

where $u_0 \in B(\Omega, H)$, $M(\omega,t) : (\Omega \times [0,T]) \times \tilde{H} \rightarrow 2^H$, is maximal monotone with $D(M) = (\Omega \times [0,T]) \times \tilde{H}$, $M(\omega,t)$ is l. s. c. from $\tilde{H}$ into $\tilde{H}$, $C : \Omega \times \tilde{H} \rightarrow \tilde{H}$ is a completely continuous, random operator with $D(C) = \Omega \times \tilde{H}$, and

$$a(\omega)(t,s) \in L^\infty(\Omega, L^2([0,T] \times [0,T]))$$

Then, since it is known that the sum of a maximal monotone and a completely continuous operator is pseudomonotone, we see that Theorem 3.1. is applicable to the existence problem of local random solutions of (*).

Remark 3.2. Since operators on H rarely satisfy the assumption that their corresponding shift-operators ($\tilde{A}$) are pseudomonotone on $\tilde{H}$, we proceed to obtain an existence result for the case that $A : \Omega \times H \rightarrow 2^H$, itself, is pseudomonotone and random.

To this goal, the following Lemma, that can be proved by a combination of Thm 3.4 of [13], Lemma 1.1, and Thm 3.5 of [8], will be useful.

Lemma 3.1. *Let $A : \Omega \times H \rightarrow 2^H$ be a random operator with $D(A) = \Omega \times H$ and nonempty closed and convex values. Let $A(\omega, \cdot)$ be l. s. c. from H into H and u. s. c. from H into H_w. Suppose moreover that A satisfies (A1). Then A is $\mathbf{A} \times \mathbf{B}_H$-measurable.*

We next state the main result of this section.

Theorem 3.2. *Let $A : \Omega \rightarrow 2^H$ be a pseudomonotone, random operator, with $D(A) = \Omega \times H$, that satisfies (A1) and (A2). Let $A(\omega, \cdot)$ be l. s. c. from H into H and u. s. c. from H into H_w. Suppose that $k \in \tilde{H}$ is continuous and satisfies (K1) and (K2). Then, for each $f \in \tilde{H}$, the inclusion (3.1) has random solution.*

Proof. By [9], (Proposition 2), and Lemma 3.1, the operators $L^*(\omega)$ and $\frac{1}{n}I + L^*(\omega)\tilde{A}(\omega)[L(\omega) + f(\omega)]$ satisfy, respectively, the assumptions on L and T of Theorem 2.1.

Hence, for each $n \geq 1$, there exists a $u_n \in B(\Omega, \tilde{H})$ such, that

$$(3.6) \qquad 0 \in L^*(\omega)u_n(\omega) + \frac{1}{n}u_n(\omega) + L^*(\omega)\tilde{A}(\omega)\left[L(\omega)u_n(\omega) + f(\omega)\right]$$

for all $\omega \in \Omega$

Since, for each $\omega \in \Omega$ $L^*(\omega)$ is injective, (3.6) implies

$$(3.7) \qquad 0 \in u_n(\omega) + \frac{1}{n}S(\omega)u_n(\omega) + L^*(\omega)\tilde{A}(\omega)\left[L(\omega)u_n(\omega) + f(\omega)\right].$$

where $S : \Omega \times \tilde{H} \to \tilde{H}$ is defined by $S(\omega)v = L^*(\omega)^{-1}v$.

As in the previous proofs we may conclude that there exists a subsequence $\{u_m(\omega)\}$ and a measurable mapping $u : \Omega \to B_M$ such that $u_m(\omega) \rightharpoonup u(\omega)$. It follows that the sequence $\{w_m\}$ defined by

$$w_m(\omega) \in L^*(\omega)\tilde{A}(\omega)[L(\omega)u_m(\omega) + f(\omega)]$$

and

$$L^*(\omega)u_m(\omega) + \frac{1}{m}u_m(\omega) = -w_m(\omega)$$

has a weak limit in $\tilde{H}$, whereby the proof can be accomplished as in Thm 2 of [9]. ∎

E x a m p l e 3.2. Let H be a real Hilbert space and $A : \Omega \times 2^H$ be a completely continuous, random operator wit $D(A) = \Omega \times H$ satisfying (A1) and (A2). Suppose that $A(\omega)u(\omega)$ is convex for each $u \in B(\Omega, H)$. Let $K \in L^\infty(\Omega, L^2(0,T))$ be continuous, such that for each $\omega \in \Omega$, $t \in [0,\infty]$, $K(\omega)t$ and $\frac{\mathrm{d}K(\omega)t}{\mathrm{d}t}$ are nonnegative and convex, and $K(\omega)0 = 0$. Then, it can easily be seen, that for each $f \in \tilde{H}$, the inclusion (3.1) has a random solution.

4. Nonlinear random Volterra equations

In this section, we study the existence and uniqueness of random solutions to the nonlinear, random Volterra equation

$$u(\omega)t + \int_0^t k(\omega)(t-s)A(\omega)u(\omega)s ds = f(\omega)t, \ \omega \in \Omega, \tag{4.1}$$

where $0 < T < \infty$

In the previous paragraph we considered the case where A is an operator from H into 2^H, which consists a restrictive hypothesis from the application viewpoint. Here, on the contrary, we study the case where A is a singlevalued operator from a real Hilbert subspace V of H into its dual V^*.

So let H be a real separable Hilbert space, V a real Hilbert space densely and continuously imbedded in H, and hence $V \subset H \subset V^*$.

Let $\tilde{V}$, H, $\tilde{V}^*$ be the spaces $L^2(\Omega, L^2(0,T;V))$, $L^2(\Omega, L^2(0,T;H))$ and $L^2(\Omega, L^2(0,T;V^*))$ respectively. The pairing between $\tilde{V}$ and $\tilde{V}^*$ is $< \cdot, \cdot >$, which coincides with the inner product in H, for u, v, $\in H$.

The norms of V, H, V^* will be denoted by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$, respectively, a symbolism which is also maintained for the norms of the spaces $\tilde{V}$, $\tilde{H}$, $\tilde{V}^*$ respectively.

We next prove the main result of this section

Theorem 4.1. *Let $A : \Omega \times V \to V^*$ be a pseudomonotone, random operator satisfying*

$$\|A(\omega)u\|_* < c_1(\omega)(1+\|u\|),\ u \in V,\ \omega \in \Omega \tag{4.2}$$

and

$$c_2(\omega)\|u\|^2 < c_3(\omega) + (A(\omega)u, u),\ u \in V,\ \omega \in \Omega \tag{4.3}$$

where $c_j : \Omega \to R^+$ satisfies $\sup c_j(\omega) < \infty$ *for $j = 1,\ 2,\ 3$.*
Let $k \in L^\infty(\Omega, L^2(0,T))$ be a continuous, random function, satisfying the hypotheses (K1) and (K2) of the previous section.
Then, for each $f \in B(\Omega, \tilde{V})$, the equation (4.1) has the solution in $B(\Omega, \tilde{V})$.

Remark 4.1. As before, to write the equation (4.1) in operator form, we introduce an operator $\tilde{A} : \Omega \times \tilde{V} \to \tilde{V}^*$ defined by $(\tilde{A}(\omega)u)(\omega, t) = A(\omega)u(\omega)t$ for each $u \in \tilde{V}$ and $\omega \in \Omega$. It can, easily , be seen that A satisfies the hypotheses

$$\|\tilde{A}(\omega)u\|_* < \tilde{c}_1(\omega)(1+\|u\|),\ u \in \tilde{V},\ \omega \in \Omega \tag{4.4}$$

and

$$\tilde{c}_2(\omega)\|u\|^2 < \tilde{c}_3(\omega) + \langle \tilde{A}(\omega)u, u \rangle,\ u \in \tilde{V},\ \omega \in \Omega \tag{4.5}$$

where $\tilde{c}_j : \Omega \to R^+$ satisfies $\sup \tilde{c}_j(\omega) < \infty$ for $j = 1,\ 2,\ 3$ and depends only on c_1, c_2, c_3 and T.

It can be seen ([10], Lemma 2), that the equation (4.1) has a solution $u \in \tilde{V}$ iff the equation

$$v + \tilde{A}(Lv + f) = 0 \tag{4.6}$$

has a solution $v \in \tilde{V}^*$.
The proof of Theorem 4.1 follows from Proposition 4.1 and a result of N. Hirano [10].

Proposition 4.1. *The equation*

$$L^*(\omega)u(\omega) + \varepsilon u(\omega) + L^*(\omega)\tilde{A}(\omega)\left[L(\omega)u(\omega) + f(\omega)\right] = 0 \tag{4.7}$$

has for any $\varepsilon > 0$ *a solution* $u \in B(\Omega, \tilde{H})$ *such that there exists a sequence* $\{v_k\} \subset B(\Omega, \tilde{V})$ *with*

$$v_k(\omega) \longrightarrow u(\omega) \text{ in } \tilde{H}, \text{ for all } \omega \in \Omega$$

and $L(\omega)v_k(\omega) \longrightarrow L(\omega)u(\omega)$ *in* $\tilde{V}$, *for all* $\omega \in \Omega$

Proof. Let D_n be the set $\{v \in \tilde{V} : |v| \leq n\}$, for each $n \geq 1$. Applying Theorem 2.1 to the operator

$$\varepsilon u(\omega) + L^*(\omega)\tilde{A}(\omega)\left[L(\omega)u(\omega) + f(\omega)\right] : \Omega \times D_n \to \tilde{V}^*$$

we can establish the existence of a weakly in $\tilde{H}$ converging subsequence $\{u_k(\omega)\}$, to $u(\omega)$, say.

By Mazur's Lemma, we get $\{u_k(\omega)\} \subset cocl_{\tilde{H}} \{u_k(\omega) : k \geq 1\}$ such that $v_k(\omega) \to u(\omega)$ in $\tilde{H}$ and $L(\omega)v_k(\omega) \to L(\omega)u(\omega)$ in $\tilde{V}$. Then, as in [10] we can prove that

$$(4.8) \quad \left\langle L^*(\omega)v - (-\varepsilon u(\omega) - L^*(\omega)\tilde{A}(\omega)[L(\omega)u(\omega) + f(\omega)]) \ , \ v - u(\omega)\right\rangle \geq 0$$

for all $v \in \tilde{V}$ and $\omega \in \Omega$.

Since $-\left(\varepsilon u(\omega) + L^*(\omega)\tilde{A}(\omega)\left[L(\omega)u(\omega) + f(\omega)\right]\right) \in \tilde{H}$ and L^* is continuous on $\tilde{H}$, (4.8) is valid for all $v \in \tilde{H}$ and $\omega \in \Omega$. Finally since L^* is maximal monotone, we get that

$$-\varepsilon u(\omega) + L^*(\omega)\tilde{A}(v)\left[L(\omega)u(\omega) + f(\omega)\right] = L^*(\omega)u(\omega). \blacksquare$$

Theorem 4.2. (uniqueness) *Let the assumptions of Theorem 4.1 hold. If, moreover,* $(A(\omega)u - A(\omega)v \ , \ u - v) \leq 0$ *implies* $A(\omega)u = A(\omega)v$ *for each* u, $v \in V$ *and* $\omega \in \Omega$, *then for each* $f \in B(\Omega, \tilde{V})$, *the equation (4.1) has a unique solution.*

The proof is based on Theorem IV 2.6 of [14].

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