Mathematatica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica
Balkanica

New series Vol.7, 1993, Fasc.3-4

Uniqueness of Regular Solutions for a Class of Nonlinear Degenerating Hyperbolic Equations*

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Using energy-integral method we prove uniqueness of regular solutions of a boundary value problem in R^2 for a class of nonlinear hyperbolic equations with degenerating principal part.

Consider the equation

$$(1) \qquad Lu = k(y)\,u_{xx} - (l(y)\,u_y)'_y = g(x,y,u)$$

in the bounded simply-connected domain $G \subset R^2$ with a piecewise-smooth boundary $\partial G = AB \cup BC \cup AC$, where

(a) k and l are continuous functions defined for $y \geq 0$, $k(0) = l(0) = 0$, $k(y) > 0$, $l(y) > 0$ for $y > 0$ and $\lim_{y \to 0} k(y)/l(y)$ exists;

(b) AB is the segment with endpoints $A = (0,0)$ and $B = (1,0)$, the lines AC and BC are parts of the characteristics $\sqrt{k}\, dy = \pm\sqrt{l}\, dx$ of equation (1), issued respectively from the points A and B, which intersect at the point $C = (1/2, d)$, $d > 0$.

Equation (1) is strictly hyperbolic for $y > 0$ and its order degenerates on the line $y = 0$. It is known (see A. V. B i t s a d z e [2]) that the classical boundary value problems are not always correctly posed for equations with degenerating principal parts. We consider the following boundary value problem:

* Supported by contracts MM–71/91 and MM–48/91 with the Ministry of Education and Science of Bulgaria.

Problem A. *Find a solution of equation (1) in G satisfying the boundary condition*

$$u|_{AC} = \varphi, \tag{2}$$

where φ is a given continuous function on AC.

Definition. A regular solution of Problem A is a function $u(x,y)$, defined and continuous in $\overline{G}$, with continuous first partial derivatives in $\overline{G} \setminus AB$ such that for $y \to 0$ $u_x, u_y = O\left(1/l^\delta\right)$, $0 \le \delta < 1$, and continuous second partial derivatives in G, which satisfies (1), (2).

Our aim in this paper is to examine the uniqueness of regular solutions of Problem A. Similar investigations for nonlinear mixed type differential equations were carried out by D. K. Gvazava [3, 4] for the Tricomi problem and by M. Saigo [6, 7] for the Frankl problem with appropriate maximum principals. M. Schneider [8] offered an energy-integral method for the uniqueness problem of regular solutions for a class of nonlinear equations of mixed type considering the Tricomi boundary value problem. By the same method A. K. Aziz, M. Schneider [1] obtained uniqueness theorems for the Frankl - Morawetz problem and J. M. Rassias [5] made an analogous research for the Tricomi problem. Our approach is also based on energy - integral considerations. An announcement of our results is given in [9].

Theorem. *If $k, l \in C^2[0,d]$ and $g, g_u(x,y,u) \in C(G \times R)$ satisfy the conditions*

(i) $(kl)' > 0$ *for* $y > 0$;

(ii) $h(y) := 1 + 2l\left(k/(kl)'\right)' > 0$ *for* $y > 0$;

(iii) $0 \le -g_u(x,y,u) \le h(y)\left[(kl)'\right]^2/4k^2 l$ *in* $G \times R$,

then Problem A has at most one regular solution.

Proof. Suppose u_1 and u_2 are two regular solutions of Problem A. Then $v = u_1 - u_2$ satisfies the linear boundary value problem

$$\begin{cases} L(v) = H(x,y)v & \text{in } G \\ v|_{AC} = 0, \end{cases}$$

where $H(x,y)$ is determined by the equality

$$g(x,y,u_1) - g(x,y,u_2) = \int_0^1 \frac{d}{dt}\left[g(x,y,u_2 + tv)\right] dt = vH(x,y),$$

i.e. $H(x,y) = \int_0^1 g_u(x,y,u_2+tv)\,dt$.

Put $d_n v = kv_x\,dy + lv_y\,dx$. Applying Green's formula to a domain G^0, $\overline{G^0} \subset G$, with piecewise - smooth boundary ∂G^0, we get the following

Lemma 1. *If $a,b,c,v \in C^2(\overline{G^0})$, then*

$$\begin{aligned}
&\iint_{G^0} 2L(v)(av+bv_x+cv_y)\,dxdy \\
&= \int_{\partial G^0} \{2(av+bv_x+cv_y)\,d_n v + \left(kv_x^2 - lv_y^2\right)(cdx-bdy) - v^2 d_n a\} \\
&- \iint_{G^0} \left(\widetilde{A}v_x^2 + 2\widetilde{B}v_xv_y + \widetilde{C}v_y^2 + \widetilde{D}v^2\right)\,dxdy,
\end{aligned}$$

where

$$\widetilde{A} = k(b_x - c_y) - bk' + 2ka, \quad \widetilde{B} = kc_x - lb_y,$$

$$\widetilde{C} = l(b_x - c_y) - 2la + cl', \quad \widetilde{D} = -ka_{xx} + l'a_y + la_{yy}.$$

Put $G_\varepsilon = \{(x,y) \in G : y > \varepsilon\}$. Using Lemma 1 for domains, which approximate from inside G_ε and passing to limits we get for $v = u_1 - u_2$ and $a,b,c \in C^2\left(\overline{G} \setminus AB\right)$ the equality

$$\begin{aligned}
&\int_{\partial G_\varepsilon} \{2(av+bv_x+cv_y)\,d_n v + \left(kv_x^2 - lv_y^2\right)(cdx-bdy) - v^2 d_n a\} \\
&= \iint_{G_\varepsilon} \left\{\widetilde{A}v_x^2 + 2\widetilde{B}v_xv_y + \widetilde{C}v_y^2 + \left(\widetilde{D} + 2Ha\right)v^2 + 2Hbv_xv + 2Hcv_yv\right\}\,dxdy.
\end{aligned} \tag{3}$$

Further we choose the functions a,b,c seeking the line integral in (3) to be nonpositive or to tend to zero as $\varepsilon \to 0$ and the expression in the double integral to be a positive semidefinite quadratic form in variables (v_x, v_y, v). Under the conditions this choice is possible we get $v = 0$.

The line integral in (3) equals the sum of line integrals on $A_\varepsilon B_\varepsilon$, $B_\varepsilon C$ and CA_ε, where A_ε and B_ε are points in which the line $y = \varepsilon$ intersects the characteristics AC and BC of equation (1).

Consider the line integral $\int_{A_\varepsilon B_\varepsilon} \omega$, where

$$\omega = 2(av+bv_x+cv_y)\,d_n v + \left(kv_x^2 - lv_y^2\right)(cdx-bdy) - v^2 d_n a.$$

We have

$$\int\limits_{A_\varepsilon B_\varepsilon} \omega = \int\limits_{A_\varepsilon B_\varepsilon} \{2\,(av + bv_x + cv_y)\,(kv_x dy + lv_y dx)$$

$$+ \left(kv_x^2 - lv_y^2\right)(cdx - bdy) - v^2\,(la_y dx + ka_x dy)\}$$

$$= \int\limits_{A_\varepsilon B_\varepsilon} \left(ckv_x^2 + 2blv_xv_y + clv_y^2\right) dx + \int\limits_{A_\varepsilon B_\varepsilon} \left(-a_y lv^2 + 2alvv_y\right) dx.$$

Since $v_y = O\left(1/l^\delta\right)$, where $0 \le \delta < 1$, we get

$$\int\limits_{A_\varepsilon B_\varepsilon} \left(-a_y lv^2 + 2alvv_y\right) dx \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

On the other hand under the conditions

$$c \le 0, \quad b^2 l - c^2 k \le 0 \ \text{ in } G,$$

the quadratic form in variables (v_x, v_y)

$$ckv_x^2 + 2blv_xv_y + clv_y^2$$

is negative semidefinite and therefore the integral

$$\int\limits_{A_\varepsilon B_\varepsilon} \left(ckv_x^2 + 2blv_xv_y + clv_y^2\right) dx$$

is nonpositive.

Consider the line integral $\int\limits_{B_\varepsilon C} \omega$. Put $\alpha = 2avd_nv - v^2 d_n a$. Since on BC $dx = -\sqrt{k/l}\,dy$ we obtain

$$\omega|B_\varepsilon C = \alpha + 2\,(bv_x + cv_y)\,(kv_x dy + lv_y dx) + \left(kv_x^2 - lv_y^2\right)(cdx - bdy)$$

$$= \alpha + 2\,(bv_x + cv_y)\left(kv_x dy - \sqrt{kl}\ v_y dy\right) + \left(kv_x^2 - lv_y^2\right)\left(-\sqrt{k/l}c - b\right) dy$$

$$= \alpha + \left(\sqrt{k}\ v_x - \sqrt{l}\ v_y\right)^2 \left(b - \sqrt{k/l}\ c\right) dy.$$

Using that on the characteristic line BC $d_n w = -\sqrt{kl}\,dw$ for any C^1-function w, we get

$$\int\limits_{B_\varepsilon C} \alpha = \int\limits_{B_\varepsilon C} \{ad_nv^2 - v^2 d_n a\} = -\int\limits_{B_\varepsilon C} \left\{a\sqrt{kl}\ dv^2 + v^2\sqrt{kl}\ da\right\}$$

$$= -\int\limits_{B_\varepsilon C} d\left(a\sqrt{kl}\ v^2\right) + \int\limits_{B_\varepsilon C} v^2\left\{d\left(a\sqrt{kl}\right) + \sqrt{kl}\ da\right\}.$$

The first integral here tends to zero as $\varepsilon \to 0$. Indeed, since $v(C) = 0$,

$$-\int\limits_{B_\varepsilon C} d\left(a\sqrt{kl}\, v^2\right) = a(B_\varepsilon)\sqrt{k(\varepsilon)\,l(\varepsilon)}\, v^2(B_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

For the second integral we obtain

$$\int\limits_{B_\varepsilon C} v^2\left\{d\left(a\sqrt{kl}\right) + \sqrt{kl}\, da\right\} = \int\limits_{B_\varepsilon C} v^2\left\{a\left(\sqrt{kl}\right)' - 2ka_x + 2\sqrt{kl}\, a_y\right\}\, dy,$$

using again that $dx = \sqrt{k/l}\, dy$ on BC.

Therefore the line integral $\int\limits_{B_\varepsilon C} \omega$ equals a sum of a term which tends to zero as $\varepsilon \to 0$ and

$$\int\limits_{B_\varepsilon C} \left\{v^2\left[a\left(\sqrt{kl}\right)' - 2ka_x + 2\sqrt{kl}\, a_y\right] + \left(\sqrt{k}\, v_x - \sqrt{l}\, v_y\right)^2\left(b - \sqrt{k/l}\, c\right)\right\}\, dy.$$

The last integral is nonpositive if on BC the conditions

$$a\left(\sqrt{kl}\right)' - 2ka_x + 2\sqrt{kl}\, a_y \le 0, \quad b - \sqrt{k/l}\, c \le 0$$

hold.

Consider the line integral $\int\limits_{CA_\varepsilon} \omega$. Since $v|_{CA} = 0$ it is easy to see that

$$d_n v = \sqrt{kl}\, dv = 0 \quad \text{and} \quad kv_x^2 - lv_y^2 = 0 \quad \text{on } CA_\varepsilon,$$

Therefore $\int\limits_{CA_\varepsilon} \omega = 0$.

From the above considerations we get immediately the following

Lemma 2. *If* $\left(\sqrt{kl}\right)' > 0$ *for* $y > 0$ *and the functions* a, b, c *satisfy*

$$a = const \le 0, \quad b = \sqrt{k/l}\, c, \quad c \le 0, \tag{4}$$

then the line integral in (3) *equals a sum of line integrals, which are either nonpositive or tend to zero as* $\varepsilon \to 0$.

Consider now the double integral in (3) and suppose a, b, c satisfy condition (4). Then by $b = \sqrt{k/l}\, c$ we get

$$\widetilde{A}\widetilde{C} - \widetilde{B}^2 = -\left[c\left(\sqrt{kl}\right)' - 2\sqrt{kl}a\right]^2,$$

hence the quadratic form $\widetilde{A}v_x^2 + 2\widetilde{B}v_xv_y + \widetilde{C}v_y^2$ is positive semidefinite only if $\widetilde{A} \geq 0$, $\widetilde{C} \geq 0$ and $\widetilde{A}\widetilde{C} - \widetilde{B}^2 = 0$. Therefore we have an unique choice $c = 4akl/(kl)'$. Taking $a = -1/2$ we get

$$\widetilde{A} = kh, \quad \widetilde{B} = \sqrt{kl}h, \quad \widetilde{C} = lh, \quad \widetilde{D} = 0,$$

where $h = 1 + 2l\left(k/(kl)'\right)' > 0$ by condition (ii). Then the quadratic form under the double integral in (3) is

$$h\left(\sqrt{k}v_x + \sqrt{l}v_y\right)^2 + 2H\frac{c}{\sqrt{l}}\left(\sqrt{k}v_x + \sqrt{l}v_y\right)v - Hv^2, \tag{5}$$

i.e. we get a quadratic form in two variables $\sqrt{k}v_x + \sqrt{l}v_y$ and v. It is easy to see that by (iii) follows

$$H \leq 0, \qquad H^2c^2/l + hH \leq 0,$$

i.e. the quadratic form (5) is positive semidefinite.

Passing to the limit as $\varepsilon \to 0$ we get that the double integral over G equals zero, hence the form (5) is zero in G. Since it equals the sum of the two squares

$$h\left(\sqrt{k}\; v_x + \sqrt{l}\; v_y + \frac{Hc}{h\sqrt{l}}v\right)^2 + \left(-H - \frac{H^2c^2}{hl}\right)v^2$$

we have

$$\sqrt{k}\; v_x + \sqrt{l}\; v_y + \frac{Hc}{h\sqrt{l}}v = 0 \quad \text{in } G. \tag{6}$$

The characteristic lines of equation (6) are determined by $dx/\sqrt{k} = dy/\sqrt{l}$, i.e. they are parallel to AC. It is evident that through any point of G passes a characteristic line of (6) issued from a corresponding point of BC.

On the other hand by (3) and Lemma 2 we obtain $-\frac{1}{2}\int\limits_{B_\epsilon C} v^2\left(\sqrt{kl}\right)'\,dy = 0$, which implies $v|_{BC} = 0$. By the theory of first order linear partial differential equations it follows that $v = 0$ in G, i.e. the regular solutions u_1 and u_2 coincide. The theorem is proved.

R e m a r k. In an analogous way it is possible to investigate uniqueness of regular solutions of the same boundary value problem when $g = g(x, y, u, u_x, u_y)$ and get corresponding sufficient conditions.

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Received 16.07.1992