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Characterization of the Upper and the Lower Classes for Diffusion Processes in Terms of Convergence rates

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For diffusion processes obtained as solutions of stochastic differential equations, we characterize the lower classes and the upper classes of functions in terms of convergence rates.

1. Introduction

Let $X(t),\ t \geq 0$ be the solution of the one-dimensional stochastic nonlinear homogenous differential equation

$$\begin{aligned} \mathrm{d}X(t) &= a(X(t))\mathrm{d}t + b(X(t))\mathrm{d}W(t),\ t \geq 0 \\ X(0) &= X_0, \end{aligned} \tag{1.1}$$

where $W(t),\ t \geq 0$ is a standerd Wiener process and the random variable X_0 with $\mathbf{E}(X_0^2) < \infty$ is independent of $\{W(t), t \geq 0\}$.

Let $\phi(t), t \geq 0$ be a non-negative and non-decreasing function which increases to infinity as $t \to \infty$. We say that $\phi(t)$ belongs to the upper class or to the lower class of the process X according as

$$\mathbf{P}\{X(t) > \phi(t),\ \text{i.o. as } t \to \infty\} = 0 \ \ \mathit{or} \ \ 1$$

(i.o. is the abbreviation of "infinitely often").

Let $X_n,\ n = 1,\ 2,\ \ldots$ be a sequence of independent random variables. W. Feller

[5] and K. L. Chung [2] have studied the asymptotic rates as $n \to \infty$ of $S_n = \sum_{i=1}^{n} X_n$ and $M_n = \max_{1 \le i \le n} |S_i|$, respectively. These papers are considered to be two fundamental papers in this area of study. W. Feller gives a brief description of the motivation of the study of upper and lower functions. He has suggested an integral criterion for a non-decreasing function $\phi(t)$ to belong to the upper class or to the lower class of S_n. Feller's technique was to calculate the tail probability and apply the Borel-Cantelli lemma for obtaining the integral criterion.

In the case of a Brownian motion (= Wiener Process), Kolmogorov (see I. Ito and H. P. McKean [9]) has developed an integral criterion so that the non-decreasing function $\phi(t)$ belongs to the upper class, or to the lower class of the Brownian motion process $\{W(t),\ t \geq 0\}$ according as a special integral converges or diverges. Problems of similar nature have been considered by V. Strassen [12] for a martingale difference sequence $\{Y_n\}$ and by N. C. Jain et al. [10] for the partial sum $S_n = \sum_{i=1}^{n} Y_i$. Note, however, that the technique followed by them is different from that of the work of Feller and Kolmogorov.

In the present paper, under certain conditions, we transform the diffusion process into another one which is close to the Brownian motion process almost surely for all sufficiently large T so that the asymptotic properties known for the Brownian motion can be applied to the sequence S_n. Several works have been published in this direction. In our previous paper, see M. N. Mishra and S. K. Acharya [11], we have developed an integral criterion to decide whether $\phi(t)$ belongs to the upper class or to the lower class for diffusion processes described by a homogeneous stochastic differential equation of Ito type.

In this paper we consider a general equation (1.1) which is a homogeneous stochastic differential equation of Ito type and have described the upper and the lower classes for the process solution in the terms of convergence rates. In article 3, Theorem 1 deals with an integral criterion to decide whether $\phi(t)$ belongs to the upper class or to the lower class of $X(t)$. J. A. Davis [3] has studied the characterization of upper and lower class functions for sequences of independent and identically distributed random variables under suitable conditions. Motivated by his work we have studied here, see Theorem 2 below, the characterization of the upper class and of the lower class functions in terms of specified convergence rates which is given in Lemma 5. Results of Lemma 2 and 5 are important for our reasoning. For readers convenience we have included their proof in the Appendix.

2. Notations and preliminaries

Consider the stochastic differential equation (1.1), where $W(t)$, $t \geq 0$ is a stanard Wiener process adapted to $\{\mathcal{F}_t, t \geq 0\}$, an increasing family of sub σ-algebras, in some basic probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Further we introduce some assumptions denoted by (A_1), (A_2), ..., etc.

(A_1) We assume that $a(x)$ and $b(x)$ are nonnegative measurable functions on $(-\infty, \infty)$ satisfying Lipschitz's condition.

Let us denote various positive constants by C_1, C_2, ... etc.

Let $h(z) = \int_0^z (1/b(y))dy$ exists and $g(z) = h^{-1}(z)$, the inverse of h. The process $\eta(t) = h(X(t))$ satisfies the equation (see I. I. Gikhman and A. V. Skorohod [7])

$$\begin{aligned} d\eta(t) = &\left[a(X(t))\, h'(X(t)) + (1/2)\, b^2(X(t))\, h''(X(t))\right] dt \\ &+ h'(X(t))\, b(X(t))\, dW(t). \end{aligned} \tag{2.1}$$

From the definition of h and the relation $X(t) = g(\eta(t))$ we obtain

$$\begin{aligned} d\eta(t) = &\{a(\, g(\eta(t)))\, /\, b(\, g(\eta(t))) \\ &- (1/2)b'(\, g(\eta(t)))\}\, dt + dW(t). \end{aligned} \tag{2.2}$$

Let us assume that

$$\begin{aligned} |a(\, g(\eta(t)))\, /\, b(\, g(\eta(t))) - (1/2)\, b'(\, g(\eta(t)))| \\ \leq C_1/\, (1 + \eta(t))^{1+\delta} \end{aligned}$$

for some $0 < \delta < 1$, with probability 1 for each t.

(A_3) Further, we assume that for some constants $\bar{b} > 0$, and $0 < \mu < 1$,

$$|\bar{b}\, h(x) - x| \leq C_2/\, (1 + |x|)^{1-\mu}$$

Using (A_1) and (A_2) it can be shown that

$$\mathbf{E}\{\, |\eta(t) - W(t)|^2\, \} \leq C_3 t^{1-\delta} + C_4 \tag{2.3}$$

Then

$$\mathbf{E}\{\, |X(t) - \bar{b}W(t)|^2\, \} \leq C_5 t^{1-\nu} + C_6 \tag{2.4}$$

where $\nu = \min(\delta, \mu)$ (see Friedman [6]).

To prove the main results we need the following lemmas.

Lemma 1. *Let $\phi(t)$, $t \geq 0$ be a non negative and non decreasing function, and $\{t_n,\ t_n \geq 1\}$ be a positive increasing sequence. Then*

$$\sum_{n=1}^{\infty} \phi(t_n)\ t_n^{-1}\ e^{-\phi^2(t_n/2)} < \infty$$

if and only if

$$\sum_{n=1}^{\infty} (\log\ \log t_n)\ (t_n \phi(t_n))^{-1} \times \exp\left[(-\phi^2(t_n)/2)\ (1 + C_7\ /\ \log\log t_n)\right] \ < \ \infty.$$

For the proof of this lemma we refer to J. A. Davis [3].

Lemma 2. *Suppose conditions (A_1), (A_2) and (A_3) hold, $X(t)$ is the solution of (1.1) and $\mathbf{E}(X_0)^2 < \infty$. Then for some constant $\bar{b} > 0$ we have*

$$|X(t) \ - \ \bar{b}\ W(t)| \ = \ o\,(t^{1/2}\ (\log\ \log t)^{-1/2}) \quad a.s.\ as\ t \to \infty.$$

The proof of this lemma is given in the Appendix.

We take the following assumption on the drift coefficeint $a(x)$:

$$(\mathrm{A}_4\text{‘}) \qquad \int_{-\infty}^{\infty} a(x)\ \mathrm{d}x \ < \ \infty$$

and for arbitrary $0 < \delta < 1$,

$$|a(x)| \ < \ C_8 /\ (1 + |x|)^{1+\delta}$$

Lemma 3. *Under the condition (A_4)*

$$\mathbf{E}\left\{ \ \left| \ X(t) - \int_0^t b(X(s))\ \mathrm{d}W(s) \ \right|^2 \ \right\} = \ O\,(t^{1-\delta})$$

(see A. F. riedman [6])

Lemma 4. *Let $\{W(t),\ t \geq 0\}$ be a stanard Wiener process and Z be a nonnegative random variable. Then for every $x \in \mathbb{R}$ and $\varepsilon > 0$,*

$$|\mathbf{P}\{\ W(Z) \ \leq \ x\ \} - \Phi(x)| \leq (\ 2\varepsilon\ )^{1/2} \ + \mathbf{P}\{\ |Z - 1| \ > \ \varepsilon\ \}$$

(see P. H. all and C. C. Heyde [8])

Lemma 5. *Suppose that there exists a positive function $Q(t) \uparrow \infty$ such that for arbitrary $0 < \delta < 1$, $t^{1-\delta}(Q(t))^{-1} \to 0$ and a.s.*

$$\frac{I(t)}{Q(t)} \to 1 \text{ as } t \to \infty, \text{ where } I(t) = \int_0^t b^2(X(s))\, ds.$$

Then for each $x \in \mathbb{R}$ and any positive function $\varepsilon(t)$, $t \geq 0$ decreasing to zero as $t \to \infty$, we have

$$\left| \mathbf{P}\left\{ \frac{X(t)}{\sqrt{Q(t)}} \leq x \right\} - \Phi(x) \right| = O\left(\varepsilon(t)^{1/2}\right). \tag{2.5}$$

The proof of this lemma is given in the Appendix.

For a sequence $t_n \uparrow \infty$ as $n \to \infty$, relation (2.5) can be written as

$$\left| \mathbf{P}\left\{ \frac{X(t_n)}{\sqrt{Q(t_n)}} \leq x \right\} - \Phi(x) \right| = O\left(\varepsilon(t_n)^{1/2}\right).$$

Therefore

$$\begin{aligned} &\sum_{n=1}^{\infty} (\log\ \log t_n)\, t_n^{-1} \left| \mathbf{P}\left\{ \frac{X(t_n)}{\sqrt{Q(t_n)}} > \phi(t_n) \right\} - \{1 - \Phi(\phi(t_n))\} \right| \\ &\leq C_9 \sum_{n=1}^{\infty} (\log\ \log t_n)\, t_n^{-1} \left(\varepsilon(t_n)^{1/2}\right) \\ &= C_9 \sum_{n=1}^{\infty} \frac{\log\ \log t_n}{t_n\, t_n^{1/5}} < \infty \quad (\text{ choosing } \varepsilon(t) = t^{-2/5}) \end{aligned} \tag{2.6}$$

Now with $\phi(t_n) \uparrow \infty$, we find

$$1 - \Phi(\phi(t_n)) \sim (2\pi)^{-1/2} (\phi(t_n))^{-1} e^{-\phi^2(t_n)/2},$$

so that from (2.6) we get

$$\begin{aligned} &(\log\ \log t_n)\, t_n^{-1} \left| \mathbf{P}\left\{ \frac{X(t_n)}{\sqrt{Q(t_n)}} > \phi(t_n) \right\} \right. \\ &\left. - C_{10}(\phi(t_n))^{-1} e^{-\phi^2(t_n)/2} \right| < \infty. \end{aligned} \tag{2.7}$$

3. Main results

Theorem 1. *Let*
(i) (A_1), (A_2) *and* (A_3) *hold,*
(ii) $X(t)$ *be a solution of (1.1) with* $\mathbf{E}X_0^2 < \infty$,
(iii) $\phi(t) > 0$ *increases monotonically to infinity with* t. *Then*

$$\mathbf{P}\{\ X(t) \ > \ t^{1/2}\ \phi(t)\ \text{i.o. as } t \to \infty\ \} \ = \ 0 \text{ or } 1$$

according to as

$$\int_1^\infty \frac{\phi(t)}{t}\ \exp(\ -\frac{\phi^2(t)}{2}\ )\ dt \ < \ \infty \text{ or } = \ \infty \tag{3.1}$$

The proof of the above Theorem follows from Theorem 3.2 of S. K. Acharya and M. N. Mishra [1]

Remark 1. From Lemma 1 and Theorem 1 it is easy to observe that for a positive monotonic increasing sequence $\{t_n\}$,

$$\sum_{n=1}^{\infty}(\log\ \log t_n)\ (t_n\phi(t_n))^{-1}\ \exp\left[(-\phi^2(t_n)/2\ (1 + C_7\ /\ \log\log t_n)\right]\ < \ \infty$$

and

$$\sum_{n=1}^{\infty}(\log\ \log t_n)\ (t_n\phi(t_n))^{-1}\ \exp\left[(-\phi^2(t_n)/2\right]\ < \ \infty$$

iff $\phi(t_n)$ is in the upper class.

Theorem 2. *Let* $X(t)$ *be the solution of the equation (1.1) with* $\mathbf{E}X_0^2 < \infty$ *and* $Q(t)$ *be a positive function as defined in Lemma 5. Then the positive increasing function* $\phi(t)$ *is in the upper class of* $X(t)$ *iff*

$$\int_1^\infty (\log\ \log t)\ t^{-1}\ \mathbf{P}\left\{\ \frac{X(t)}{\sqrt{Q(t)}}\ > \ \phi(t)\right\} dt \ < \ \infty$$

Proof. Let $\phi(t)$ be in the upper class. Then by the integral test

$$\int_1^\infty \frac{\phi(t)}{t}\ e^{-\phi^2(t)/2}\ dt \ < \ \infty,$$

i.e. there exists a sequence $\{t_n\} \uparrow \infty$ such that

$$\sum_{n=1}^{\infty} \frac{\phi(t_n)}{t_n} e^{-\phi^2(t_n)/2} < \infty,$$

which implies that

$$\sum_{n=1}^{\infty} (\log\ \log t_n)\ t_n^{-1} \phi^{-1}(t_n)\ e^{-\phi^2(t_n)/2} < \infty$$

(by Lemma 2). Therefore, from (2.7) we get

$$\sum_{n=1}^{\infty} (\log\ \log t_n)\ t_n^{-1}\ \mathbf{P}\left\{ \frac{X(t_n)}{\sqrt{Q(t_n)}} > \phi(t_n) \right\} < \infty$$

and hence

$$\int_1^{\infty} (\log\ \log t)\ t^{-1}\ \mathbf{P}\left\{ \frac{X(t)}{\sqrt{Q(t)}} > \phi(t) \right\} \mathrm{d}t < \infty$$

Again let

$$\int_1^{\infty} (\log\ \log t)\ t^{-1}\ \mathbf{P}\left\{ \frac{X(t)}{\sqrt{Q(t)}} > \phi(t) \right\} \mathrm{d}t < \infty$$

i.e. for some sequence $t_n \uparrow \infty$ as $n \to \infty$, we have

$$\sum_{n=1}^{\infty} (\log\ \log t_n)\ t_n^{-1}\ \mathbf{P}\left\{ \frac{X(t_n)}{\sqrt{Q(t_n)}} > \phi(t_n) \right\} < \infty$$

Then by relation (2.7), we find

$$\sum_{n=1}^{\infty} (\log\ \log t_n)\ t_n^{-1} \phi^{-1}(t_n)\ e^{-\phi^2(t_n)/2} < \infty,$$

i.e.

$$\sum_{n=1}^{\infty} (\log\ \log t_n)\ t_n^{-1} \phi^{-1}(t_n)$$
$$\times \exp\left[(-\phi^2(t_n)/2)\ (1 + C_7\ /\ \log\log t_n)\right] < \infty$$

Therefore by Lemma 1

$$\sum_{n=1}^{\infty} \phi(t_n)\, t_n^{-1}\, e^{-\phi^2(t_n)/2} < \infty$$

which implies that

$$\int_1^{\infty} \phi(t)\, t^{-1}\, e^{-\phi^2(t)/2} < \infty$$

Hence $\phi(t)$ is in the upper class of the process $X(t)$.

By the relation (2.7) the convergence or divergence of the series

$$\sum_{n=1}^{\infty} (\log\, \log t_n)\, t_n^{-1} \phi^{-1}(t_n)\, e^{-\phi^2(t_n)/2}$$

implies the same property for the series

$$\sum_{n=1}^{\infty} (\log\, \log t_n)\, t_n^{-1}\, \mathbf{P}\left\{ \frac{X(t_n)}{\sqrt{Q(t_n)}} > \phi(t_n) \right\}$$

Thus a result is analogous to Borel zero-one law can be stated as follows.

Proposition. *Let $X(t)$ be the solution of equation (1.1) with $\mathbf{E}X_0^2 < \infty$. Then for a positive monotonic increasing function $\phi(t)$,*

$$\mathbf{P}\left\{ X(t) > t^{1/2}\phi(t) \text{ i.o.} \right\} = 0 \text{ or } 1.$$

according as

$$\int_1^{\infty} (\log\, \log t)\, t^{-1}\, \mathbf{P}\left\{ \frac{X(t)}{\sqrt{Q(t)}} > \phi(t) \right\} dt < \infty \quad \text{or} = \infty.$$

4. Appendix

Proof of Lemma 2. Under the conditions of the lemma, we have from (2.4)

$$\mathbf{E}\{\, |X(t) - \bar{b}W(t)|^2 \,\} \le C_5 t^{1-\nu} + C_6$$

Now for $t_m = m^{\lambda}$, $\lambda = 4/\delta$, m a positive integer, we have

$$\mathbf{P}\left\{ \sup_{t_m \le t \le t_{m+1}} \left| \frac{X(t) - \bar{b}\, W(t)}{t^{1/2}\, (log\ logt)^{-1/2}} \right| > \frac{1}{m} \right\}$$

$$\le \frac{\mathbf{E}\{|X(t) - \bar{b}\, W(t)|^2\}\ m^2}{t_m\ (log\ logt_{m+1})^{-1}}$$

$$\le \frac{C_5 t^{1-\nu} + C_6}{t_m\ (log\ logt_{m+1})^{-1}} < \frac{C_{11}\ \log m}{m^2}.$$

Now since $\sum_{m=1}^{\infty} C_{11}(\log m)/m^2 < \infty$, we have, by applying Borel-Cantelli Lemma, that

$$\mathbf{P}\left\{ \sup_{t_m \le t \le t_{m+1}} \left| \frac{X(t) - \bar{b}\, W(t)}{t^{1/2}\, (log\ logt)^{-1/2}} \right| > \frac{1}{m} \quad \text{i.o.} \right\} = 0$$

and consequently,

$$\mathbf{P}\left\{ \lim_{t \to \infty} \left| \frac{X(t) - \bar{b}\, W(t)}{t^{1/2}\, (log\ logt)^{-1/2}} \right| = 0 \right\} = 1$$

This completes the proof of Lemma 2.

Proof of Lemma 5. Let us consider now the stochastic integral

$$\int_0^t b(X(s))\, \mathrm{d}W(s).$$

It is easy to see that it is a square integrable, zero mean martingale. By Theorem 2.3 in P. D. Feigin [4] (which is due to H.Kunita and S. Watanabe) there exists a standard Brownian motion, say $B(t), t \ge 0$, such that

$$\int_0^t b(X(S))\, \mathrm{d}W(s) = B(I_t)$$

where the "new time" I_t is given by

$$I_t = \int_0^t b^2(X(S))\, \mathrm{d}s$$

Therefore by Lemma 3,

$$\mathbf{E}\{\ |X(t) - B(I_t)|^2\ \} = O\,(t^{1-\delta}). \tag{4.1}$$

Now, for any $\varepsilon > 0$, using Kolmogorov's inequality we have

$$\lim_{t\to\infty} \mathbf{P}\left\{ \sup_{t\geq 0}\left| \frac{X(t)}{\sqrt{Q(t)}} - \frac{B(I_t)}{\sqrt{Q(t)}} \right| \geq \varepsilon \right\}$$
$$\leq \lim_{t\to\infty} \mathbf{E}\left\{ \left| \frac{X(t)}{\sqrt{Q(t)}} - \frac{B(I_t)}{\sqrt{Q(t)}} \right|^2 \right\} /\varepsilon^2$$
$$= \lim_{t\to\infty} \frac{1}{Q(t)} \mathbf{E}\{ |X(t) - B(I_t)|^2 \} /\varepsilon^2$$
$$\leq \lim_{t\to\infty} \frac{C_{12}t^{1-\delta}}{Q(t)\varepsilon^2} \quad \text{(by Lemma 3)} = 0$$

Hence

$$\frac{X(t)}{\sqrt{Q(t)}} = \frac{B(I_t)}{\sqrt{Q(t)}} \quad \text{a.s. as } t \to \infty. \tag{4.2}$$

So

$$\left| \mathbf{P}\left\{ \frac{X(t)}{\sqrt{Q(t)}} \leq x \right\} - \Phi(x) \right|$$
$$= \left| \mathbf{P}\left\{ \frac{B(I_t)}{\sqrt{Q(t)}} \leq x \right\} - \Phi(x) \right|$$
$$= \left| \mathbf{P}\left\{ B\left(\frac{I_t}{Q(t)}\right) \leq x \right\} - \Phi(x) \right|$$
$$\leq (2\,\varepsilon(t))^{1/2} + \mathbf{P}\left\{ \left| \frac{I_t}{Q(t)} - 1 \right| \geq \varepsilon \right\} \quad \text{(by Lemma 4)}$$
$$= O(\varepsilon(t))^{1/2}.$$

Thus Lemma 5 is also proved.

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