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Alexander's Operator for Sequences

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In this paper we define two weak types of starshaped sequences. One of them shows a close connection between starshaped and superadditive sequences, while the other one is used for the determination of linear operators which conserve some sequence classes. We obtain so a discrete operator of Alexander type.

1. Introduction

Sequences with some special properties can occur in many unexpected branches. For example, if the positive sequence $(a_n)_{n\geq 1}$ has the property:

$$(k+1)\,a_{k+1} \leq k\,a_k, \quad \forall k \geq 1,$$

then the complex function f defined by $f(z) = z + a_2 z^2 + \ldots \quad (a_1 = 1)$ is close-to-convex and a similar condition implies that f is a starlike function (see [6]). Also convex, quasiconvex and other sequences are used in the theory of Fourier series (see [3] for many references), giving conditions for summability.

In this paper we deal with more classes of sequences. The following sets are well known (see for example [4]): the set of convex sequences:

$$K = \{(x_n)_{n\geq 0} : \; x_{n+2} - 2x_{n+1} + x_n \geq 0, \; \forall n \geq 0\}$$

and also that of superadditive sequences:

$$S = \{(x_n)_{n\geq 0} : \; x_{n+m} + x_0 \geq x_n + x_m, \; \forall n, m \geq 1\}.$$

In [7] we have considered the set of starshaped sequences:

$$S^\star = \{(x_n)_{n\geq 0} : \ (x_n - x_0)/n \leq (x_{n+1} - x_0)/(n+1), \ \ \forall n \geq 1\}$$

proving also that:

$$K \subset S^\star \subset S. \tag{1}$$

Then we have used in [9] a weaker form of superadditivity introducing the set:

$$W = \{(x_n)_{n\geq 0} : \ x_{n+1} + x_0 \geq x_n + x_1, \ \ \forall n \geq 1\}.$$

Here we define also two weaker kinds of starshapedness and establish their relations with the previous notions.

In [10] there are characterized the weighted arithmetic means that preserve the convexity. We have obtained a simpler characterization in [8] and then we have proved that it is also valid for the preservation of the starshapedness or the superadditivity (see [9]). In what follows we want to determinate all the linear positive operators of another special type which conserve one of the above properties. Thus we get a discrete operator which resembles Alexander's integral operator used in the theory of complex functions (see for example [5]).

2. Weakly starshaped sequences

Define the following two sets of sequences:

$$J^\star = \{(x_n)_{n\geq 0} : \ x_{nm} - x_0 \geq n(x_m - x_0), \ \ \forall n, m \geq 1\}$$

and

$$V^\star = \{(x_n)_{n\geq 0} : \ x_n - x_0 \geq n(x_1 - x_0), \ \ \forall n \geq 1\}.$$

The first of them can be considered as a Jensen starshapedness and the second as a very weak kind of starshapedness. We have obviously:

$$S^\star \subset J^\star \subset V^\star$$

but we want to combine it with (1). Before doing this we add also the set of linear (or zero) sequences:

$$Z = \{(x_n)_{n\geq 0} : \ \exists a, b \in R, \ x_n = an + b, \ \forall n \geq 0\}.$$

Lemma 1. *The following inclusions:*

$$Z \subset K \subset S^\star \subset S \subset W$$
$$\cap \quad \cap \tag{2}$$
$$J^\star \subset V^\star$$

hold.

Proof. The inclusions $S \subset J^\star$ and $W \subset V^\star$ can be proved by mathematical induction. The other relations are in (1) or are obvious.

Remark 1. The inclusions:

$$S^\star \subset S \subset J^\star$$

show a close connection between starshapedness and superadditivity. In fact a superadditive sequence verifies even a stronger inequality than that used in the definition of $J^\star$, namely:

$$x_n - x_0 \geq [n/m](x_m - x_0), \quad n \geq m,$$

where $[x]$ denotes the integer part of x.

We have given in [7] a representation formula for sequences from K : a sequence $(x_n)_{n\geq 0}$ belongs to k if and only if

$$x_n = \sum_{k=0}^{n}(n-k+1)y_k, \quad \text{with } y_k \geq 0 \text{ for } k \geq 2.$$

Also we have used representation formulas for sequences from $S^\star$ and in [9] for those from W. We add here such formulas for sequences from S, $J^\star$ and $V^\star$. Each of them is easy to verify.

Lemma 2. *Every sequence $(x_n)_{n\geq 0}$ can be represented by:*

$$x_n = n\sum_{k=1}^{n} z_k - (n-1)z_0, \quad \textit{for } n \geq 0. \tag{3}$$

It belongs to:

i) $S^\star$ *if and only if*

$$z_k \geq 0 \textit{ for } k \geq 2; \tag{4}$$

ii) S *if and only if*

$$n\sum_{k=n+1}^{n+m} z_k + m\sum_{k=m+1}^{m+n} z_k \geq 0, \quad for\ n,m \geq 1;$$

iii) $J^\star$ *if and only if*

$$\sum_{k=n+1}^{nm} z_k \geq 0, \quad for\ n \geq 1, m \geq 2; \tag{5}$$

iv) W *if and only if*

$$n\, z_{n+1} + \sum_{k=2}^{n+1} z_k \geq 0, \quad for\ n \geq 1;$$

v) $V^\star$ *if and only if*

$$\sum_{k=n+1}^{2n} z_k \geq 0, \quad for\ n \geq 1.$$

Remark 2. For sequences from W and $V^\star$ we have also simpler representations:

$$x_n = \sum_{k=2}^{n} w_k + nx_1 - (n-1)x_0, \quad \text{with } w_k \geq 0, \text{ for } k \geq 2, \tag{6}$$

respectively:

$$x_n = v_n + nx_1 - (n-1)x_0, \quad \text{with } v_n \geq 0, \text{ for } n \geq 2, \tag{7}$$

both valid for $n \geq 2$.

3. Linear operators

Let $Q = (q_{nm})_{0\leq m\leq n}$ be a strictly positive triangular matrix. For a sequence $x = (x_n)_{n\geq 0}$ consider the associated sequence $L^Q(x)$ defined by:

$$L_n^Q(x) = \sum_{k=0}^{n} q_{nk}\, x_k, \quad \forall n \geq 0.$$

We get so a linear operator L^Q defined on the space of all real sequences with values in the same space. It is also isotonic, that is $L^Q(x)$ is positive if x is positive. Given a set X of sequences, an usual problem is to characterize the matrices Q with the property that X is invariant under L^Q, that is $L^Q(X) \subset X$. We have such characterizations for the set K of convex sequences (see [1] and [2]). We have also the following general result:

Lemma 3. *If one of the sets $K, S^\star, S, W, J^\star$ or $V^\star$ is invariant under L^Q, then Z is also invariant with respect to it.*

Proof. Let x be an arbitrary sequence from Z. If the set

$$X \in \{K, S^\star, S, W, J^\star, V^\star\}$$

is invariant under L^Q, as $x \in X$, we have $L^Q(x) \in X$. By (2) we get $L^Q(x) \in V^\star$. But $-x$ also belongs to Z, which gives $L^Q(x) \in Z$.

In what follows we want to give explicitly the matrices Q with the property that Z is invariant under L^Q, supposing Q of some special types. We begin with the case of weighted arithmetic means studied in [8] and [10]. Let $p = (p_n)_{n\geq 0}$ be a strictly positive sequence. For any sequence $x = (x_n)_{n\geq 0}$ we define the sequence $A^p(x)$ of weighted arithmetic means of x by:

$$A_n^p(x) = \frac{\sum_{k=0}^n p_k x_k}{\sum_{k=0}^n p_k}, \quad \forall n \geq 0. \tag{8}$$

We note that one can define a matrix Q using p by:

$$q_{nm} = \frac{p_m}{\sum_{k=0}^n p_k}, \quad 0 \leq m \leq n.$$

Lemma 4. *The inclusion:*

$$A^p(Z) \subset Z \tag{9}$$

is valid if and only if there is an $u \geq 0$ such that:

$$p_n = p_0 \binom{u+n-1}{n}, \quad \forall n \geq 0, \tag{10}$$

where

$$\binom{v}{0} = 1, \quad \binom{v}{n} = \frac{v(v-1)\ldots(v-n+1)}{n!}, \quad n \geq 1.$$

Proof. If (9) holds, we must have $a, b \in R$ such that:

$$\frac{\sum_{k=0}^n k\, p_k}{\sum_{k=0}^n p_k} = an + b, \quad \forall n \geq 0. \tag{11}$$

For $n = 0$ we get $b = 0$ and $n = 1$ gives $a = p_1/(p_0 + p_1)$ so that (11) becomes:

$$\sum_{k=0}^n k\, p_k = \frac{n\, p_1 \sum_{k=0}^n p_k}{p_0 + p_1}, \quad \forall n \geq 0. \tag{12}$$

Thus, by subtraction, we have:

$$(n+1)\,p_{n+1} = (p_1/p_0)\sum_{k=0}^{n} p_k.$$

Denoting $p_1/p_0 = u$, we obtain, again by subtraction:

$$p_{n+1} = \frac{p_n(u+n)}{n+1}$$

which gives (10).

Conversely, if p_n is given by (10) then (9) is valid, because (12) means $A^p(z) = (u/(u+1))\,z$, where $z = (n)_{n\geq 0}$.

The second case which we study is obtained by putting $q_{nk} = p_k$. Thus we have the linear operator B^p defined by:

$$B_n^p(x) = \sum_{k=0}^{n} p_k x_k, \quad \forall n \geq 0.$$

We denote by:

$$Z_0 = \{(x_n)_{n\geq 0} : \exists a, x_n = an, \ \forall n \geq 0\}.$$

Lemma 5. i) *There is no sequence p with property:*

$$B^p(Z) \subset Z; \tag{13}$$

ii) *The operator B^p satisfies the inclusion:*

$$B^p(Z_0) \subset Z_0$$

if and only if

$$p_n = p_1/n, \quad \forall n \geq 1.$$

Proof. To obtain (13) it is necessary and sufficient that for arbitrary a and b there exist constants c, d, e and f such that:

$$\sum_{k=0}^{n} a\,k\,p_k = c\,n + d, \quad \sum_{k=0}^{n} b\,p_k = e\,n + f, \quad n \geq 0.$$

For $n = 0$ we have $d = 0$, $bp_0 = f$ and for $n \geq 1$: $anp_n = c$, $bp_n = e$. Since $b \neq 0$ leads to a contradiction, we must have $b = 0$, $e = 0$, $c = ap_1$ and $p_n = p_1/n$.

Remark 3. Taking $p_1 = 1$, we get an operator which we denote simply by B, thus:

$$B_n(x) = \sum_{k=1}^{n} x_k/x, \quad \forall n \geq 1. \tag{14}$$

As we pointed out in the introduction, this operator resembles Alexander's integral operator.

4. A hierarchy of starshapedness

In what follows we want to investigate the sufficiency of the previous conditions. First, we denote:

$$M^u T = \{x : A^p(x) \in T\},$$

where T is an arbitrary set of sequences and A^p is given by (8) with p taken as in (10). We have proved in [8] and [9] that:

$$\begin{array}{ccccccc} K \subset M^u K \subset & S^\star & \subset & S & \subset W & & \\ & \cap & & \cap & & & \\ & M^u S^\star & \subset & M^u S & \subset M^u W & & \end{array} \tag{15}$$

that is, the condition is sufficient for the sets $K, S^\star$ and W. We try to extend this result by taking into account (2). But, as in the case of the set S, we are not able to prove the inclusion $J^\star \subset M^u J^\star$ because the representation given by (3) and (5) for the sequences of $J^\star$ is too complicated.

Lemma 6. *For every $u \geq 0$ the inclusion:*

$$V^\star \subset M^u V^\star$$

is valid.

Proof. Let $x = (x_n)_{n \geq 0}$ be an arbitrary sequence of $V^\star$. It may be represented as in (7) by:

$$x_n = v_n + nx_1 - (n-1)x_0$$

with $v_0 = v_1 = 0$ and $v_n \geq 0$ for $n \geq 2$. So:

$$\begin{aligned} A_n^u(x) &= \sum_{k=0}^{n} \frac{\binom{u+k-1}{k} x_k}{\binom{u+n}{n}} \\ &= \sum_{k=0}^{n} \frac{\binom{u+k-1}{k} v_k}{\binom{u+n}{n}} + (x_1 - x_0)nu/(u+1) + x_0 \\ &= w_n + n\, A_1^u(x) - (n-1)\, A_0^u(x), \end{aligned}$$

where $w_0 = w_1 = 0$ and $w_n \geq 0$ for $n \geq 2$, that is $A^u(x) \in V^\star$.

To use the operator B given by (14), taking into account Lemma 5, we must use only sequences which have the first item zero. So, for a given set T of sequences, we denote by:

$$T_0 = \{x = (x_n)_{n\geq 0},\ x \in T,\ x_0 = 0\}$$

its subset with desired property. Also we denote:

$$M^0 T_0 = \{x : B(x) \in T_0\}.$$

We get the following characterizations:

Lemma 7. *The sequence $(x_n)_{n\geq 1}$ belongs to:*
i) $M^0 K_0$, *iff* $x_{n+1}/(n+1) \geq x_n/n$, *for* $n \geq 2$;
ii) $M^0 S_0^\star$, *iff* $\sum_{k=1}^{n} (x_n/n - x_k/k) \geq 0$ *for* $n \geq 2$;
iii) $M^0 W_0$, *iff* $x_n/n \geq x_1$ *for* $n \geq 2$;
iv) $M^0 V_0^\star$, *iff* $\sum_{k=2}^{n} (x_k/k - x_1) \geq 0$ *for* $n \geq 2$.

Proof. We have only to compute:
i) $B_{n+2}(x) - 2B_{n+1}(x) + B_n(x) = x_{n+2}/(n+2) - x_{n+1}/(n+1)$;
ii) $B_{n+1}(x)/(n+1) - B_n(x)/n = \frac{nx_{n+1}/(n+1) - \sum_{k=1}^{n} x_k/k}{n(n+1)}$;
iii) $B_{n+1}(x) - B_n(x) - B_1(x) = x_{n+1}/(n+1) - x_1$;
iv) $B_n(x) - nB_1(x) = \sum_{k=1}^{n} x_k/k - nx_1 = \sum_{k=1}^{n} (x_k/k - x_1)$.

Lemma 8. *The inclusions*

$$\text{(16)} \quad \begin{array}{ccccccccc} K_0 & \subset & S_0^\star & \subset & S_0 & \subset & W_0 & \subset & V_0^\star \\ \cap & & \cap & & & & & & \\ M^0K_0 & \subset & M^0S_0^\star & \subset & M^0S_0 & \subset & M^0W_0 & \subset & M^0V_0^\star \end{array}$$

hold.

Proof. The inclusions from the first and the second lines follow from (2), while $S_0^\star \subset M^0K_0$ and $V_0 \subset M^0W_0$ are proved in assertions i) respectively iii) of Lemma 7.

Remark 4. It is easy to see that a superadditive sequence satisfies the condition ii) of Lemma 7 for $n = 2, 3, 4, 5$, so we conjecture that:

$$S_0 \subset M^0 S_0^\star.$$

To the contrary, $W_0 \not\subset M^0 S_0$. For example, the sequence x given by $x_1 = 0$, $x_n = 1$ for $n \geq 2$, belongs to W_0, but not to $M^0 S_0$.

Also we can combine the diagrams (15) and (16).

Lemma 9. *For every $u > 0$ the inclusions:*

$$\begin{array}{ccccccc} M^u S_0^\star & \subset M^u S_0 \subset & M^u W_0 & \subset & M^u V_0^\star \\ \cap & & \cap & & \cap \\ M^0 S_0^\star & \subset M^0 S_0 \subset & M^0 W_0 & \subset & M^0 V_0^\star \end{array}$$

hold.

Proof. i) If $x = (x_n)_{n\geq 0} \in M^u S_0^\star$, then using (3) and (4) we can represent $A^u(x)$ by:

$$A_n^u(x) = n \sum_{k=1}^{n} z_k, \quad \text{with } z_k \geq 0 \text{ for } k \geq 2.$$

But then, as in [9], we have:

$$x_n = (1 + \frac{n}{u}) A_n^u(x) - \frac{n}{u} A_{n-1}^u(x), \tag{17}$$

hence

$$x_n = n \left((n-1) z_n + (u+1) \sum_{k=1}^{n} z_k \right) / u.$$

We deduce that:

$$\frac{x_n + 1}{n+1} - \frac{1}{n} \sum_{k=1}^{n} \frac{x_k}{k} = (1 + \frac{n+1}{u}) z_{n+1} + \sum_{k=2}^{n} \frac{k-1}{u} z_k \geq 0,$$

that is $x \in M^0 S_0^\star$.

ii) If $x \in M^u W_0$, we have by (6):

$$A_n^u(x) = \sum_{k=2}^{n} w_k + n w_1, \quad \text{with } w_k \geq 0 \text{ for } k \geq 2$$

if $n \geq 2$ and $A_1^u(x) = w_1$. Then $x_1 = (1 + 1/u)\, w_1$ and for $n \geq 2$, from (17):

$$x_n = n w_1 + \sum_{k=2}^{n} w_k + \frac{n}{u}(w_1 + w_n) \geq n(1 + 1/u) w_1 = n\, x_1,$$

that is $x \in M^0 W_0$.

iii) For $x \in M^u V_0^\star$ we have from (7) $A_1^u(x) = v_1$ and:

$$A_n^u(x) = v_n + n v_1, \quad \text{with } v_n \geq 0 \text{ for } n \geq 2.$$

So $x_1 = (1 + 1/u) v_1$ and from (17):

$$x_n = (1 + \frac{n}{u})\, v_n - \frac{n}{u}\, v_{n-1} + n\,(1 + \frac{1}{u})\, v_1.$$

Thus:

$$\sum_{k=2}^{n} (x_k/k - x_1) = v_n/u + \sum_{k=2}^{n} v_k/k \geq 0$$

and by Lemma 7, $x \in M^0 V_0^\star$.

We summarize the above results in the following:

Theorem. *For arbitrary $u \geq 0$ we have the inclusions:*

$$\begin{array}{ccccccccccccc}
K_0 & & \subset & & S_0^\star & \subset & S_0 & \subset & W_0 & \subset & V_0^\star & & \\
 & & & \rotatebox{-45}{$\supset$} & \rotatebox{45}{$\subset$} & & & & \rotatebox{45}{$\subset$} & \rotatebox{-45}{$\supset$} & \rotatebox{45}{$\subset$} & & \\
\cap & & M^0 K_0 & \subset & M^0 S_0^\star & \subset & M^0 S_0 & \subset & M^0 W_0 & \subset & M^0 V_0^\star & & \\
 & \rotatebox{45}{$\cup$} & & & \rotatebox{45}{$\cup$} & & & & \rotatebox{45}{$\cup$} & & \rotatebox{45}{$\cup$} & & \\
M^u K_0 & & \subset & & M^u S_0^\star & \subset & M^u S_0 & \subset & M^u W_0 & \subset & M^u V_0^\star & &
\end{array}$$

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