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On the $|\overline{N}, p_n|_k$ summability Factors of Infinite Series

Hüseyin Bor, M. Ali Sarigöl

Presented by P. Kenderov

In this paper a theorem of Bor and Sarigöl [2] on $|\overline{N}, p_n|_k$ summability factors has been proved under weaker conditions.

1. Definition. Let $\sum a_n$ be given infinite series with the sequence of partial sums (s_n). Let (p_n) be a sequence of positive real constants such that

$$P_n = \sum_{v=0}^{n} p_v \longrightarrow \infty, \text{ as } n \longrightarrow \infty, \ (P_{-i} = p_{-i}, i \geq 1) \tag{1.1}$$

The sequence to sequence transformation

$$H_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v. \tag{1.2}$$

defines the sequence of the $(\overline{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} | H_n - H_{n-1} |^k < \infty$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), then $|\overline{N}, p_n|_k$ summability is the same as $| C, 1 |_k$ (resp. $|\overline{N}, p_n|$) summability.

2. Quite recently Bor and Sarigöl [2] proved the following theorem.

Theorem A. *Let*

$$\lambda_m = o(1), \text{ and } \sum_{n=1}^{m} nP_n|\Delta^2\lambda_n| = O(1) \text{ as } m \to \infty \tag{2.1}$$

$$\frac{1}{p_m} = o(m) \text{ as } m \to \infty. \tag{2.2}$$

$$\sum_{n=1}^{m} p_n|t_n|^k = O(P_m) \text{ as } m \to \infty, \tag{2.3}$$

where (t_n) *is the n-th* $(C,1)$ *mean of the sequence* (na_n), *then the series* $\sum a_n\lambda_n$ *is summable* $|\overline{N}, p_n|_k$, $k \geq 1$.

3. The object of this paper is to prove the Theorem A under weaker conditions. Now, we shall prove the following theorem.

Theorem. *Let* (λ_n) *and* (p_n) *be two sequences such that*

$$\lambda_m \longrightarrow 0 \tag{3.1}$$

$$\sum_{v=1}^{m} (vP_v)^k (|\Delta\lambda_v| + |\Delta\lambda_{v+1}|)^{k-1} |\Delta(|\Delta\lambda_v|)| = O(1), \tag{3.2}$$

$$\sum_{v=1}^{m} P_v^k (|\lambda_v| + |\lambda_{v+1}|)^{k-1} |\Delta\lambda_v| = O(1), \tag{3.3}$$

and

$$\frac{1}{p_m} = O(m) \text{ as } m \longrightarrow \infty. \tag{3.4}$$

If

$$\sum_{v=1}^{m} p_v|t_v|^k = O(P_m) \text{ as } m \to \infty,$$

where (t_n) *is the same as in Theorem A, then the series* $\sum a_n\lambda_n$ *is summable* $|\overline{N}, p_n|_k$, $k \geq 1$.

It may be noted that condition (2.1) implies conditions (3.2) and (3.3). But the converse of this implication need not be true. So we extend the theorem A weakening the conditions. This can be shown like this: We have

$$\sum_{v=1}^{m}(vP_v)^k(|\Delta\lambda_v| + |\Delta\lambda_{v+1}|)^{k-1}|\Delta(|\Delta\lambda_v|)|$$
$$\leq \sum_{v=1}^{m}(vP_v(|\Delta\lambda_v| + |\Delta\lambda_{v+1}|))^{k-1} vP_v|\Delta^2\lambda_v|$$
$$= O(1)\sum_{v=1}^{m} vP_v|\Delta^2\lambda_v| = O(1) \text{ as } m \to \infty, \text{ by (2.1)},$$

because

$$vp_v|\Delta\lambda_v| \leq \sum_{n=v}^{\infty} nP_n|\Delta^2\lambda_n| = O(1)$$

and similarly $vp_v|\Delta\lambda_{v+1}| = O(1)$ as $v \to \infty$.
Also since

$$P_v|\lambda_v| \leq \sum_{n=v}^{\infty} P_n|\Delta\lambda_n| \leq \sum_{n=1}^{\infty} P_n \sum_{r=n}^{\infty}|\Delta^2\lambda_r|$$
$$= \sum_{r=1}^{\infty}|\Delta^2\lambda_r|\sum_{n=1}^{r} P_n \leq \sum_{v=1}^{\infty} vP_v|\Delta^2\lambda_v| < \infty$$

and similarly $P_v|\lambda_{v+1}| = O(1)$ as $v \to \infty$, we have that

$$\sum_{v=1}^{m} P_v^k(|\lambda_v| + |\lambda_{v+1}|)^{k-1}|\Delta\lambda_v| \leq \sum_{v=1}^{m}(P_v(|\lambda_v| + |\lambda_{v+1}|))^{k-1} P_v|\Delta\lambda_v|$$
$$= O(1)\sum_{v=1}^{m} P_v|\Delta\lambda_v| = O(1)\sum_{v=1}^{m} vP_v|\Delta^2\lambda_v| = O(1) \text{ by (2.1)}.$$

To show the converse of these implications it is sufficient to take as $P_n = 2^{n-1}/n^2$ for $n \geq 3$, $\lambda_n = 1/2^n$ and $k > 1$.

4. We need the following lemma for the proof of our theorem.

Lemma. *If conditions (3.1)–(3.3) of the theorem are satisfied, then*

$$nP_n| \Delta\lambda_n | = O(1) \tag{4.1}$$

$$P_n| \lambda_n | = O(1) \tag{4.2}$$

$$\sum_{v=1}^{m} v^k P_v| \Delta(P_v^{k-1}) || \Delta\lambda_{v+1} |^k = O(1) \tag{4.3}$$

and

$$\sum_{v=1}^{m} P_{v+1}^k| \Delta(v^k) || \lambda_{v+1} |^k = O(1) \text{ as } m \to \infty \tag{4.4}$$

Proof. By considering conditions (3.1) and (3.2) we have

$$(nP_n| \Delta\lambda_n |)^k = (nP_n)^k \sum_{v=n}^{\infty} \Delta(| \Delta\lambda_v |^k) \leq \sum_{v=n}^{\infty}(vP_v)^k| \Delta(| \Delta\lambda_v |^k) |.$$

On the other hand since

$$\Delta(| \Delta\lambda_v |^k) = | \Delta\lambda_v |^k - | \Delta\lambda_{v+1} |^k = kc_v^{k-1}(| \Delta\lambda_v | - | \Delta\lambda_{v+1} |)$$

where c_v lies between $| \Delta\lambda_v |$ and $| \Delta\lambda_{v+1} |$, and since $c_v \leq | \Delta\lambda_v | + | \Delta\lambda_{v+1} |$, we obtain

$$| \Delta(| \Delta\lambda_v |^k) | \leq k(| \Delta\lambda_v | + | \Delta\lambda_{v+1} |)^{k-1}| \Delta(| \Delta\lambda_v |) |.$$

So that it follows that $nP_n| \Delta\lambda_n | = O(1)$. Similarly, using the conditions (3.1) and (3.3), we have that $P_n| \lambda_n | = O(1)$.
Again, since

$$0 \leq P_{v+1}^{k-1} - P_v^{k-1} = (k-1)\xi_v^{k-2}p_{v+1}$$

where $P_v < \xi_v < P_{v+1}$, we get that $0 \leq P_{v+1}^{k-1} - P_v^{k-1} = O(1)P_{v+1}^{k-2}p_{v+1}$ for all $k \geq 2$. Thus,

$$\sum_{v=1}^{m} v^k P_v| \Delta(P_v^{k-1}) || \Delta\lambda_{v+1} |^k = \sum_{v=1}^{m} v^k P_v(P_{v+1}^{k-1} - P_v^{k-1})| \Delta\lambda_{v+1} |^k$$

$$= O(1)\sum_{v=1}^{m} v^k p_{v+1} P_{v+1}^{k-1}| \Delta\lambda_{v+1} |^k == O(1)\sum_{v=1}^{\infty} v^k p_{v+1} P_{v+1}^{k-1} \sum_{n=v}^{\infty} | \Delta(| \Delta\lambda_{n+1} |^k) |$$

$$= O(1)\sum_{n=1}^{\infty} | \Delta(| \Delta\lambda_{n+1} |^k) | \sum_{v=1}^{n} v^k p_{v+1} P_{v+1}^{k-1}$$

$$= O(1)\sum_{n=1}^{\infty}(n+1)^k P_{n+1}^k| \Delta(| \Delta\lambda_{n+1} |^k) |$$

$$= O(1)\sum_{n=1}^{\infty}(nP_n)^k(| \Delta\lambda_n | + | \Delta\lambda_{n+1} |)^{k-1}| \Delta(| \Delta\lambda_n |) | = O(1) \text{ by (3.2)},$$

and by (4.5) we have, for $1 \leq k < 2$,

$$P_{v+1}^{k-1} - P_v^{k-1} = O(1) P_v^{k-2} p_{v+1}. \tag{4.5}$$

So, for $1 \leq k < 2$,

$$\sum_{v=1}^{m} v^k P_v | \Delta(P_v^{k-1}) || \Delta\lambda_{v+1} |^k = O(1) \sum_{v=1}^{m} v^k P_v^{k-1} p_{v+1} | \Delta\lambda_{v+1} |^k$$

$$= O(1) \sum_{v=1}^{\infty} (v+1)^k P_v^{k-1} p_{v+1} \sum_{n=v}^{\infty} | \Delta(| \Delta\lambda_{n+1} |^k) |$$

$$= O(1) \sum_{v=1}^{\infty} v^k P_v^k (| \Delta\lambda_v | + | \Delta\lambda_{v+1} |)^{k-1} | \Delta(| \Delta\lambda_v |) | = O(1) \text{ by (3.2).}$$

Similarly (4.4) can be shown.

5. Proof of the Theorem. Let (T_n) be the sequence of $(\overline{N}, p_n)$ mean of the series $\sum a_n \lambda_n$. Then by the deffinition, we have that

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r \lambda_r = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v \lambda_v.$$

Then, for $n \geq 1$, we have that

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} v^{-1} P_{v-1} v \, a_v \lambda_v.$$

Applying Abel's transformation, we have that

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta(\frac{1}{v} P_{v-1} \lambda_v) \sum_{r=1}^{v} r a_r + \frac{1}{n P_n} p_n \lambda_n \sum_{v=1}^{n} v a_v$$

$$= -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v \lambda_v \frac{v+1}{v} t_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta\lambda_v \frac{v+1}{v} t_v$$

$$+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \lambda_{v+1} \frac{1}{v} t_v + \frac{1}{n P_n} (n+1) p_n \lambda_n t_n$$

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{say.}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$(5.1)\qquad \sum_{n=1}^{\infty}(P_n/p_n)^{k-1}\mid T_{n,r}\mid^k < \infty, \quad \text{for } r=1,2,3,4.$$

Now, applying the Hölder's inequality, we have that

$$\sum_{n=2}^{m+1}(P_n/p_n)^{k-1}\mid T_{n,1}\mid^k = \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left|\sum_{v=1}^{n-1}p_vt_v\lambda_v\frac{1}{v}(v+1)\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{v=1}^{n-1}p_v\mid t_v\mid^k\mid\lambda_v\mid^k\right\}\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{n-1}p_v\mid t_v\mid^k\mid\lambda_v\mid^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} = O(1)\sum_{v=1}^{m}\frac{1}{P_v}\mid\lambda_v\mid^k p_v\mid t_v\mid^k$$

$$= O(1)\sum_{v=1}^{m-1}\mid(\frac{1}{P_v}\mid\Delta\lambda_v\mid^k)\mid\sum_{r=1}^{v}p_r\mid t_r\mid^k + O(1)\frac{1}{P_m}\mid\lambda_m\mid^k\sum_{v=1}^{m}p_v\mid t_v\mid^k$$

$$= O(1)\sum_{v=1}^{m-1}\mid(\frac{1}{P_v}\mid\Delta\lambda_v\mid^k)\mid P_v + O(1)\mid\lambda_m\mid^k = O(1)\sum_{v=1}^{m-1}\mid\Delta(\mid\lambda_v\mid^k)\mid$$

$$+ O(1)\sum_{v=1}^{m-1}(\frac{p_{v+1}}{P_{v+1}}\mid\lambda_{v+1}\mid^k + O(1)\mid\lambda_m\mid^k = O(1)\sum_{v=1}^{m-1}(\mid\lambda_v\mid+\mid\lambda_{v+1}\mid)^{k-1}\mid\Delta\lambda_v\mid$$

$$+ O(1)\sum_{v=1}^{m}(\frac{p_v}{P_v}\mid\lambda_v\mid^k + O(1)\mid\lambda_m\mid^k = O(1)\sum_{v=1}^{m-1}(\mid\lambda_v\mid+\mid\lambda_{v+1}\mid)^{k-1}\mid\Delta\lambda_v\mid$$

$$+ O(1)\sum_{v=1}^{m-1}\mid\Delta(\mid\lambda_v\mid^k)\mid\sum_{r=1}^{v}\frac{p_r}{P_r} + O(1)\mid\lambda_m\mid^k\sum_{r=1}^{v}\frac{p_r}{P_r} + O(1)\mid\lambda_m\mid^k$$

$$= O(1)\sum_{v=1}^{m-1}(\mid\lambda_v\mid+\mid\lambda_{v+1}\mid)^{k-1}\mid\Delta\lambda_v\mid + O(1)\sum_{v=1}^{m-1}P_v(\mid\lambda_v\mid+\mid\lambda_{v+1}\mid)^{k-1}\mid\Delta\lambda_v\mid$$

$$+ O(1)P_m\mid\lambda_m\mid^k + O(1)\mid\lambda_m\mid^k = O(1) \text{ as } m\to\infty, \text{ by (3.1), (3.3) and (4.2).}$$

Again,

$$\sum_{n=2}^{m+1}(P_n/p_n)^{k-1}|\ T_{n,2}\ |^k = O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}P_v|\ \Delta\lambda_v\ ||\ t_v\ |\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}vP_v|\ \Delta\lambda_v\ |p_v|\ t_v\ |\right\}^k \quad (\text{ by } (3.4)\ )$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}(vP_V)^k|\ \Delta\lambda_v\ |^kp_v|\ t_v\ |^k\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m}(vP_v)^k|\ \Delta\lambda_v\ |^kp_v|\ t_v\ |^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}v^kP_v^{k-1}|\ \Delta\lambda_v\ |^kp_v|\ t_v\ |^k = O(1)\sum_{v=1}^{m-1}v^kP_v^{k-1}|\ \Delta(|\ \Delta\lambda_v\ |^k)\ |P_v$$

$$+ O(1)\sum_{v=1}^{m-1}|\ \Delta\lambda_{v+1}\ |^k|\ \Delta(v^kP_v^{k-1})\ |P_v + O(1)(mP_m|\ \Delta\lambda_m\ |)^k$$

$$= O(1)\sum_{v=1}^{m-1}(vP_v)^k|\ \Delta(|\ \Delta\lambda_v\ |^k)\ | + O(1)\sum_{v=1}^{m-1}|\ \Delta\lambda_{v+1}\ |^kv^k|\ \Delta(P_v^{k-1})\ |P_v$$

$$+ O(1)\sum_{v=1}^{m-1}|\ \Delta\lambda_{v+1}\ |^kP_{v+1}^k|\ \Delta(v^k)\ | + O(1)(mP_m|\ \Delta\lambda_m\ |)^k$$

$$= O(1)\sum_{v=1}^{m-1}(vP_v)^k(|\ \Delta\lambda_v\ | + |\ \Delta\lambda_{v+1}\ |)^{k-1}|\ \Delta(|\ \Delta\lambda_v\ |)\ |$$

$$+ O(1)\sum_{v=1}^{m-1}v^k|\ \Delta(P_v^{k-1})\ ||\ \Delta\lambda_{v+1}\ |^kP_v$$

$$+ O(1)\sum_{v=1}^{m-1}P_{v+1}^k|\ \Delta(v^k)\ ||\ \Delta\lambda_{v+1}\ |^k + O(1)(mP_m|\ \Delta\lambda_m\ |)^k = O(1),$$

as $m \to \infty$, by virtue of lemma and (3.2).

Considering (3.4), we have that

$$\sum_{n=2}^{m+1}(P_n/p_n)^{k-1}\mid T_{n,3}\mid^k=\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left|\sum_{v=1}^{n-1}P_v\mid\lambda_{v+1}\mid\frac{1}{v}t_v\right|^k$$

$$=O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}P_v\mid\lambda_{v+1}\mid p_v\mid t_v\mid\right\}^k$$

$$=O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v^k\mid\lambda_{v+1}\mid^k p_v\mid t_v\mid^k\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}$$

$$=O(1)\sum_{v=1}^{m}(P_v\mid\lambda_{v+1}\mid)^k p_v\mid t_v\mid^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}$$

$$=O(1)\sum_{v=1}^{m}P_v^{k-1}\mid\lambda_{v+1}\mid^k p_v\mid t_v\mid^k$$

$$=O(1)\sum_{v=1}^{m-1}\mid\Delta(P_v^{k-1}\mid\lambda_{v+1}\mid^k)\mid\sum_{r=1}^{v}p_r\mid t_r\mid^k$$

$$+O(1)P_m^{k-1}\mid\lambda_{m+1}\mid^k\sum_{r=1}^{m}p_r\mid t_r\mid^k$$

$$=O(1)\sum_{v=1}^{m-1}\mid\Delta(P_v^{k-1}\mid\lambda_{v+1}\mid^k)\mid P_v+O(1)(P_m\mid\lambda_{m+1}\mid)^k$$

$$=O(1)\sum_{v=1}^{m-1}P_v^k\mid\Delta(\mid\lambda_{v+1}\mid^k)\mid$$

$$+O(1)\sum_{v=1}^{m-1}\mid\lambda_{v+2}\mid^k\mid\Delta(P_v^{k-1})\mid P_v+O(1)(P_{m+1}\mid\lambda_{m+1}\mid)^k$$

$$=O(1)\sum_{v=1}^{m-1}P_{v+1}^k\mid\Delta(\mid\lambda_{v+1}\mid)^k\mid+O(1)\sum_{v=1}^{m-1}P_v\mid\Delta(P_v^{k-1})\mid\mid\lambda_{v+2}\mid^k$$

$$+O(1)(P_{m+1}\mid\lambda_{m+1}\mid)^k=O(1)\sum_{v=1}^{m}P_v^k(\mid\lambda_v\mid+\mid\lambda_{v+1}\mid)^{k-1}\mid\Delta\lambda_v\mid$$

$$+O(1)\sum_{v=1}^{m-1}P_v\mid\Delta(P_v^{k-1})\mid\mid\lambda_{v+2}\mid^k+O(1)(P_{m+1}\mid\lambda_{m+1}\mid)^k$$

On the other hand, by following the way of the proof of the lemma and considering (4.5) and (4.6), it follows that

$$\sum_{v=1}^{m-1} P_v | \Delta(P_v^{k-1}) || \lambda_{v+2} |^k = O(1), \text{ as } m \to \infty \text{ by (3.3).}$$

So that,

$$\sum_{n=2}^{m} (P_n/p_n)^{k-1} | T_{n,3} |^k = O(1), \text{ as } m \to \infty \text{ by (3.3) and (4.2).}$$

Finally, as in $T_{n,1}$, we have that

$$\sum_{n=1}^{m} (P_n/p_n)^{k-1} | T_{n,4} |^k = O(1), \sum_{n=1}^{m} (p_n/P_n) | \lambda_n |^k | t_n |^k = O(1), \text{ as } m \to \infty.$$

This completes the proof of the theorem.

References

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Department of Matematics
Erciyes University
Kayseri 38039
TURKEY

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