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Asymptotic Laws for a Class of Diffusion Processes

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Let $X(t)$ be a time homogeneous solution of the one-dimensional Itô stochastic differential equation. We study upper and lower functions for the modulus of continuity of $X(t)$ as well as for large increments. Moreover, a Darling-Erdös type extreme value limiting law is derived.

1. Introduction

Let $X(t)$ be the solution of the one-dimensional Itô stochastic differential equation

(1) $$dX(t) = b(X(t))\,dt + dW(t)$$

with initial condition

(2) $$X(0) = X_0,$$

where $W(t)$, $t \geq 0$, is a standard Wiener process and X_0 is independent of $\mathcal{F}\{W(t),\ t \geq 0\}$ with $EX_0^2 < \infty$.

We shall assume throughout that $b(x)$ is a real-valued function, well-defined and measurable for $x \in (-\infty, \infty)$, and satisfying the following conditions:

(A1) For some constants K and $0 < \delta < 1$,

(3) $$|b(x)| \leq \frac{K}{(1+|x|)^{1+\delta}}.$$

(A2) For $c > 0$ and $x, y \in (-\infty, \infty)$ there exists a constant L_c such that

$$|b(x) - b(y)| \leq L_c |x - y|, \tag{4}$$

where $|x| \leq c,\ |y| \leq c$.

Under conditions (A1) and (A2) and the initial condition $X(0) = X_0$, I. I. Gihman and A. V. Skorohod [6] have shown that there exists a unique solution $X(t)$ of (1) in an arbitrary time interval $[0, T]$, and that

$$X(t) = X_0 + \int_0^t b(X(s))\, ds + W(t). \tag{5}$$

S. K. Acharya and M. N. Mishra [1],[12] investigated the law of iterated logarithm as well as upper and lower functions for the solution $X(t)$. For earlier results in this direction we refer to A. Friedman [5].

In this paper our aim is to derive further asymptotic properties of $X(t)$ as given in (5). In section 2 we investigate the modulus of continuity for the solution $X(t)$ on $[0, T]$. Section 3 is devoted to the study of large increments of $X(t)$ on $[0, T]$ as $T \to \infty$. Finally, in section 4 we prove a Darling-Erdös type extreme value limiting law. The main tool in the proofs is to derive an embedding of $X(t)$ to the standard Wiener process $W(t)$ with a suitable rate.

2. The modulus of continuity of $X(t)$

The modulus of continuity of the Wiener process has been studied by P. Lévy [10], [11]. It reads as follows (cf. e.g. M. Csörgö and P. Révész [3], Theorem 1.1.1):

Theorem 2A. *Let $W(t)$, $0 \leq t \leq 1$, be a standard Wiener process on $[0, 1]$. Then*

$$\lim_{h \to 0} \sup_{0 \leq t \leq 1-h} |W(t+h) - W(t)| \,/\, (2h \log(1/h))^{1/2} \overset{a.s.}{=} 1 \tag{6}$$

and

$$\lim_{h \to 0} \sup_{0 \leq t \leq 1-h} \sup_{0 \leq s \leq h} |W(t+s) - W(t)| \,/\, (2h \log(1/h))^{1/2} \overset{a.s.}{=} 1. \tag{7}$$

From equation (5) we immediately have, for $0 \leq t < T - h$, $0 \leq s \leq h$,

$$X(t+s) - X(t) = \int_t^{t+s} b(X(u))\, du + W(t+s) - W(t). \tag{8}$$

By assumption (A1), and $0 \le s \le h$,

$$\left| \int_t^{t+s} b(X(u))\, du \right| \le Kh. \tag{9}$$

On the other hand, for any fixed $T > 0$,

$$\{T^{-1/2} W(sT),\ 0 \le s \le 1\} \stackrel{\mathcal{D}}{=} \{W(s),\ 0 \le s \le 1\}. \tag{10}$$

On combining (8)–(10), we have proved the following result:

Theorem 2.1. *Let $X(t)$, $0 \le t \le T$, be the solution of (1) and (2). Under assumptions (A1) and (A2), we have*

$$\lim_{h \to 0} \sup_{0 \le t \le T-h} |X(t+h) - X(t)| \,/\, (2h \log(1/h))^{1/2} \stackrel{a.s.}{=} 1 \tag{11}$$

and

$$\lim_{h \to 0} \sup_{0 \le t \le T-h} \sup_{0 \le s \le h} |X(t+s) - X(t)| \,/\, (2h \log(1/h))^{1/2} \stackrel{a.s.}{=} 1. \tag{12}$$

Our next object is to obtain an integral test criterion corresponding to the asymptotics of (11) and (12). Let H_ϵ be the class of functions $h(\cdot)$ on $(0, \epsilon)$ to $[0, \infty)$ such that $h(t) \uparrow \infty$ as $t \downarrow 0$ and $t^{1/2}h(t) \downarrow 0$ as $t \downarrow 0$. We can divide the class of functions H_ϵ into upper and lower class functions as follows:

Definition 2.1. A function $h(\cdot) \in H_\epsilon$ belongs to the upper class, say U_ϵ, if , given $t > 0$, for almost every ω there exists $\delta > 0$ such that if $u \ge 0, v \ge 0, 0 < u + v < \delta$, then $|X(t+v) - X(t-u)| < 2^{1/2}(u+v)^{1/2}h(u+v)$. V_ϵ is the complement of U_ϵ in H_ϵ.

In order to derive upper and lower functions for $X(t)$ we need the following result for the Wiener process $W(t)$:

Theorem 2B. *Let $h \in H_\epsilon$. Then*

$$h \in \tilde{U}_\epsilon \iff I(h) = \int_{0+}^{\epsilon} x^{-1} h^3(x) e^{-h^2(x)}\, dx < \infty, \tag{13}$$

where $\tilde{U}_\epsilon$ denotes the analogue of U_ϵ in case of $W(t)$.

The above result is due to N.C. Jain and S.J. Taylor [8], stated there for d-dimensional Brownian motion.

By the arguments leading to Theorem 2.1 we have seen that, given $t > 0$,

$$|X(t+v) - X(t-u) - (W(t+v) - W(t-u))| \le K(v+u). \tag{14}$$

By the latter estimate, the following theorem is obtained:

Theorem 2.2 *Let $X(t)$ be the solution of (1) and (2), and let $h \in H_\epsilon$. Then*

$$h \in U_\epsilon \iff I(h) = \int_{0+}^{\epsilon} x^{-1} h^3(x) e^{-h^2(x)}\, dx < \infty.$$

Proof. Let $h \in H_\epsilon$ be such that $I(h) < \infty$. In view of relation (14), and the fact that $x^{1/2}h(x) \downarrow 0$ as $x \downarrow 0$, we estimate for given $t > 0$ as follows:

$$\begin{aligned} & |X(t+v) - X(t-u)| \geq 2^{1/2}(u+v)^{1/2} h(u+v) \\ \Longrightarrow & |W(t+v) - W(t-u)| \geq 2^{1/2}(u+v)^{1/2} h(u+v) - K(u+v) \\ & = 2^{1/2}(u+v)^{1/2}\left(h(u+v) - 2^{-1/2}K(u+v)^{1/2}\right) \\ & \geq 2^{1/2}(u+v)^{1/2}\left(h(u+v) - 1/h(u+v)\right), \end{aligned}$$

if $0 < u+v < \delta$ and $\delta \downarrow 0$. One easily checks that

$$I(h) < \infty \Longrightarrow I(h - \frac{1}{h}) < \infty.$$

Hence, $h - \frac{1}{h}$ being an upper class function for the Wiener process $\{W(t)\}$ by Theorem 2B results in h being an upper class function for $\{X(t)\}$, i.e. $h \in U_\epsilon$.

Now, consider $h \in H_\epsilon$ with $I(h) = \infty$. By a similar argument as above, we have for $0 < u+v < \delta \downarrow 0$,

$$\begin{aligned} & |W(t+v) - W(t-u)| \geq 2^{1/2}(u+v)^{1/2}(h(u+v) + 1/h(u+v)) \\ \Longrightarrow & |W(t+v) - W(t-u)| \geq 2^{1/2}(u+v)^{1/2} h(u+v) + K(u+v) \\ \Longrightarrow & |X(t+v) - X(t-u)| \geq 2^{1/2}(u+v)^{1/2} h(u+v). \end{aligned}$$

Also, $h + \frac{1}{h} \in H_\epsilon$ and

$$I(h) = \infty \Longrightarrow I(h + \frac{1}{h}) = \infty.$$

Hence follows that $h \in V_\epsilon$, i.e.

$$I(h) < \infty \iff h \in U_\epsilon.$$

Remark 2.1. The above result can be extended to higher-dimensional diffusions, since a corresponding version of Theorem 2B is available in N. C. Jain and S. J. Taylor [8].

3. Large increments of $X(t)$

M. Csörgö and P. Révész [2] have studied the problem of how large the increments of a Wiener process $\{W(t)\}$ over subintervals of length a_T of the interval $[0,T]$ can be when $T \to \infty$ and a_T is a non-decreasing function of T. Their main result is as follows (cf. e.g. Theorem 1.2.1 in M. Csörgö and P. Révész [3]):

Theorem 3A. *Let a_T $(T > 0)$ be a monotonically non-decreasing function of T for which*
(i) $0 < a_T \le T$,
(ii) T/a_T *is monotonically non-decreasing.*
Then,

$$\overline{\lim_{T\to\infty}} \sup_{0\le t\le T-a_T} \beta_T |W(t+a_T) - W(t)| \overset{a.s.}{=} 1, \tag{15}$$

$$\overline{\lim_{T\to\infty}} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \beta_T |W(t+s) - W(t)| \overset{a.s.}{=} 1, \tag{16}$$

where $\beta_T = (2a_T\{\log(T/a_T) + \log\log T\})^{-1/2}$.
If we also have
(iii) $\lim_{T\to\infty}(\log(T/a_T)/\log\log T) = \infty$,
then $\overline{\lim}_{T\to\infty}$ *in (15) and (16) can be replaced by* $\lim_{T\to\infty}$.

Here we want to derive a similar result for the diffusion process $\{X(t)\}$ of (5). Consider

$$X(t+s) - X(t) = \int_t^{t+s} b(X(u))du + W(t+s) - W(t) \tag{17}$$

for $t \in [0,T]$, $s \in [0,a_T]$. The following lemmas are needed:

Lemma 3.1. *Under condition (A1) and*

$$\int_{-\infty}^{\infty} b(x)dx = 0, \tag{A3}$$

we have

$$E\left|\int_0^t b(x(s))ds\right|^2 = O(t^{1-\delta}). \tag{18}$$

as $t \to \infty$.

For the proof of this lemma we refer to A. Friedman [5], pp. 188–189.

From Lemma 3.1 we are able to derive the following estimate on the random drift coefficient of $X(t+s) - X(t)$:

Lemma 3.2. *Let $\tau_t = \inf\{s \geq t : X(s) = 0\}$. Assume that*

$$\text{(A4)} \qquad \sup_{0 \leq t \leq T} |\tau_t - t| \overset{a.s.}{=} O\left(a_T^{(1-\delta)p}\right) \quad \text{as } T \to \infty,$$

where a_T is as in Theorem 3A, $0 < \delta < 1$ as in assumption (A1), and $p > 1/2$. Then, under conditions (A1) and (A3),

$$\int_t^{t+s} b(X(u))du \overset{a.s.}{=} O\left(a_T^{(1-\delta)p}\right) \tag{19}$$

uniformly in $t \in [0, T]$, $s \in [0, a_T]$, as $T \to \infty$.

Proof. We have

$$|\int_t^{t+s} b(X(u))\,du| \leq K|\tau_t - t| + |\int_{\tau_t}^{t+s} b(X(u))\,du|.$$

In view of assumption (A4), we only have to estimate

$$|\int_{\tau_t}^{t+s} b(X(u))\,du| = |\int_0^{t+s-\tau_t} b(\tilde{X}(u))\,du|, \tag{20}$$

where $\{\tilde{X}(u)\}$ is the unique solution of (1) with $\{\tilde{X}(0)\} = 0$, and the strong Markov property of $\{X(u)\}$ has been used.

Now, for $t_m \leq t \leq t_{m+1}$, $t_m = m^\lambda$, λ suitably chosen below, we have

$$P\left(|\int_0^t b(X(u))\,du| \geq t_m^{(1-\delta)p}\right) \leq \frac{E|\int_0^t b(X(u))\,du|^2}{t_m^{2p(1-\delta)}} = O(t_m^{(1-\delta)(1-2p)}).$$

A choice of $\lambda > 0$ such that $\lambda(1-\delta)(2p-1) > 1$ gives

$$\sum_m P\left(\sup_{t_m \leq t \leq t_{m+1}} |\int_0^t b(X(u))\,du| \geq t_m^{(1-\delta)p}\right) < \infty.$$

Since $t_{m+1}/t_m \to 1$ as $m \to \infty$, the Borel-Cantelli lemma implies

$$\int_0^t b(X(u))\,du \overset{a.s.}{=} O(t^{(1-\delta)p}). \tag{21}$$

A combination of (A4), (20) and (21) completes the proof of (19).

Making use of Lemma 3.2 we immediately obtain the following theorem:

Theorem 3.1. *Let a_T satisfy the assumptions of Theorem 3A. Assume conditions (i) and (ii) together with*

(iv) $\beta_T a_T^{(1-\delta)p} \to \infty$ as $T \to \infty$,

where β_T is as given in Theorem 3A, and where $0 < \delta < 1$ and $p > 1/2$ are as in Lemma 3.2. Under assumptions (A1)–(A4), we have

$$\overline{\lim_{T\to\infty}} \sup_{0\le t\le T-a_T} \beta_T |X(t+a_T) - X(t)| \overset{a.s.}{=} 1, \tag{22}$$

$$\overline{\lim_{T\to\infty}} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \beta_T |X(t+s) - X(t)| \overset{a.s.}{=} 1. \tag{22}$$

If we also assume condition (iii) of Theorem 3A, then $\overline{\lim}_{T\to\infty}$ in (22) and (23) can be replaced by $\lim_{T\to\infty}$.

Proof. The proof is immediate from (17), (19), (iv) and the corresponding results for $\{W(t)\}$.

Next we are going to study upper and lower class functions of the (so-called) "large increments" processes

$$Y_1(T, a_T) = a_T^{-1/2} \sup_{0\le t\le T-a_T} |X(t+a_T) - X(t)|, \tag{24}$$

$$Y_2(T, a_T) = a_T^{-1/2} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |X(t+s) - X(t)|. \tag{25}$$

Definition 3.1. a) The function h belongs to the upper-upper class of the process X ($h \in UUC(X)$) if, with probability one, there exist a (random) t_0 such that $X(t) < h(t)$ for all $t > t_0$.

b) The function h belongs to the upper-lower class of X ($h \in ULC(X)$) if, with probability one, there exists a (random) sequence $0 < t_1 < t_2 < \ldots$ with $t_i \to \infty$ as $i \to \infty$ such that $X(t_i) > h(t_i)$ for $i = 1, 2, \ldots$.

Throughout let H denote the class of functions h which are continuous, non-decreasing and satisfy $h(t) \to \infty$ as $t \to \infty$.

Let $\tilde{Y}_1(T, a_T), \tilde{Y}_2(T, a_T)$ be defined as in (24), (25) but with $\{W(t)\}$ replacing $\{X(t)\}$. J. Ortega and M. Wschebor [13] obtained the following partial description of the UUC and ULC of $\tilde{Y}_1, \tilde{Y}_2$.

Theorem 3B. *Let X be either $\tilde{Y}_1$ or $\tilde{Y}_2$. Then,*

$$I_1(h) = \int_1^\infty \frac{h(t)}{a_t} \exp\{-\frac{1}{2}h^2(t)\}\, dt = \infty \implies h \in ULC(X), \tag{26}$$

$$I_3(h) = \int_1^\infty \frac{h^3(t)}{a_t} \exp\{-\frac{1}{2}h^2(t)\}\, dt < \infty \implies h \in UUC(X).$$

Remark 3.1. J. Ortega and M. Wschebor [13] have also shown that, under additional regularity conditions on a_t, it is possible to replace $I_3(h)$ in (27) by $I_1(h)$, and hence to obtain a complete description of UUC.

In order to derive the corresponding analogue of Theorem 3B for $\{X(t)\}$, we need the following lemma (cf. N.C. Jain, K. Jogdeo and W.F. Stout [7], Lemma 2.3).

Lemma 3.3. *Let g be an eventually non-increasing function from $[1,\infty)$ to $[0,\infty)$ and f be a measurable function from $[1,\infty)$ to $[0,\infty)$. For $h \in H$, define*

$$F(h) = \int_1^\infty g(h(t))\, f(t)\, dt$$

which may be either finite or infinite. Assume that

(a_1) for every $h \in H$ and for every $A > 1$,

$$\int_1^A g(h(t))\, f(t)\, dt \;<\; \infty.$$

(a_2) there exist h_1, h_2, two members of H such that $h_1 \le h_2$, $F(h_1) = \infty$ and

$$\lim_{A\to\infty} g(h_1(A)) \int_1^A f(t)\, dt \;=\; \infty.$$

Define

$$\hat{h} \;=\; \min[\max(h, h_1),\, h_2].$$

Then for $h \in H$

(b_1) $F(h) < \infty$ implies $\hat{h} \le h$ near ∞ and $F(\hat{h}) < \infty$,

(b_2) $F(h) = \infty$ implies that $F(\hat{h}) = \infty$.

Now we are in a position to prove the following result:

Theorem 3.2. *Let X be either Y_1 or Y_2 of (24) or (25). Under the assumptions of Theorem 3.1, replacing (iv) by*

(v) $a_T^{(1-\delta)p-(1/2)} (\log(T/a_T) + \log\log T)^{1/2} \;\to\; 0$ as $T \to \infty$,

we have

$$I_1(h) = \infty \Longrightarrow h \in ULC(X), \qquad (28)$$
$$I_3(h) < \infty \Longrightarrow h \in UUC(X).$$

Proof. We consider first the upper class functions. So let $h \in H$ be such that $I_3(h) < \infty$. Let us assume that

$$h_1(t) \le h(t) \le h_2(t) \quad \text{for all } t \text{ sufficiently large}, \qquad (30)$$

where $h_1(t) = (\log(T/a_T)+\log\log T)^{1/2}$ and $h_2(t) = 2(\log(T/a_T)+\log\log T)^{1/2}$. One easily verifies that

$$I_1(h_2) \le I_3(h_2) < \infty,$$
$$I_3(h_1) \ge I_1(h_1) = \infty.$$

Let us first establish (29) for h satisfying (30). We will then show that the theorem is true for an arbitrary function $h \in H$.

By Lemma 3.2 and (5), for some $c > 0$ and $i = 1, 2$,

(31) $$|Y_i(T, a_T) - \tilde{Y}(T, a_T)| < c a_T^{(1-\delta)p-(1/2)}.$$

By assumption (v),

(32) $$a_T^{(1-\delta)p-(1/2)} \le 1/h(T) \text{ as } T \to \infty.$$

So, by (31) and (32), as $T \to \infty$

$$Y_i(T, a_T) > h(T) \implies \tilde{Y}_i(T, a_T) > h(T) - \frac{1}{h(T)}.$$

It is obvious that $I_3(h) < \infty$ implies

$$I_3(h - \frac{1}{h}) < \infty.$$

Hence by Theorem 3B, $h - \frac{1}{h} \in UUC(\tilde{Y}_i)$, and this implies $h \in UUC(Y_i)$.

Now let us remove the restriction (30) and consider an arbitrary $h \in H$. Define

$$\hat{h}(t) = \min[\max(h(t), h_1(t)), h_2(t)].$$

Then, by Lemma 3.3,

$$I_3(h) < \infty \implies I_3(\hat{h}) < \infty \text{ and } \hat{h} \le h \text{ near infinity.}$$

Again, we have $h_1(t) \le \hat{h}(t) \le h_2(t)$. Therefore,

$$P(Y_i(T, a_T) > \hat{h}(T) \text{ i.o. as } T \to \infty) = 0,$$

when $I_3(h) < \infty$. But $\hat{h}(T) \le h(T)$ near infinity. Hence

$$P(Y_i(T, a_T) > h(T) \text{ i.o. as } T \to \infty) = 0,$$

when $I_3(h) < \infty$.

Now let $h \in H$ satisfy (30) and $I_1(h) = \infty$. By a similar argument as above, for $i = 1, 2$,

$$\tilde{Y}_i(T, a_T) > h(T) + \frac{1}{h(T)}$$
$$\Longrightarrow Y_i(T, a_T) > h(T).$$

Again, also $I_1(h + 1/h) = \infty$. Hence, by Theorem 3B, $h + 1/h \in ULC(\tilde{Y}_i)$, which results in $h \in ULC(Y_i)$.

Next again by Lemma 3.3, with $\hat{h}$ as defined above,

$$I_1(h) = \infty \Longrightarrow I_1(\hat{h}) = \infty.$$

So,

$$P(Y_i(T, a_T) > \hat{h}(T) \text{ i.o. as } T \to \infty) = 1.$$

This implies that there exists a sequence $\{T_n\} \uparrow \infty$ such that

(33) $$Y_i(T_n, a_{T_n}) > \hat{h}(T_n) \text{ a.s. for every } n.$$

Since $I_1(h_2) < \infty$, we have

(34) $$Y_i(T_n, a_{T_n}) \leq h_2(T_n) \text{ a.s. for every large } n.$$

Now from (33) and (34), we get

$$\hat{h}(T_n) \leq h_2(T_n) \text{ for large } n,$$

and hence, by definition of $\hat{h}$,

$$\hat{h}(T_n) \leq h_2(T_n) \text{ for large } n,$$

i.e. $\hat{h}(T_n) \geq h(T_n)$ for large n.

Therefore by (33)

$$Y_i(T_n, a_{T_n}) > h(T_n) \text{ a.s. for large } n.$$

Hence for $I_1(h) = \infty$,

$$P(Y_i(T_n, a_{T_n}) > h(T) \text{ i.o. as } T \to \infty) = 1.$$

This completes the proof (in which we followed the technique adopted by N. C. Jain, K. Jogdeo and W. F. Stout [7]).

Remark 3.2. By Theorem 4 in J. Ortega and M. Wschebor [13], it is also possible to replace $I_3(h)$ in (29) by $I_1(h)$ under additional regularity conditions on a_t.

4. The Darling-Erdös theorem for $X(t)$

In Sections 2 and 3 we have demonstrated that a number of strong invariance principles are available for $X(t)$. The aim of this section is to show that also convergence in distribution results can be derived for $X(t)$ via their corresponding analogues for $W(t)$. As a particular example we provide the following Darling-Erdös type extreme value limiting behaviour. For a recent discussion of the latter theorem in a martingale setup we refer to U. Einmahl and D. M. Mason [4] (cf. also the references mentioned therein).

Theorem 4.1. *Let $X(t)$ be as in (5). Under conditions (A1),(A2) and (A3), we have, for fixed $\epsilon > 0$ and for all real x,*

$$\lim_{T\to\infty} P\left(a_T \sup_{\epsilon\le t\le T}(X(t)/t^{1/2}) - b_T \le x\right) = \exp(-e^{-x}), \tag{35}$$

and

$$\lim_{T\to\infty} P\left(a_T \sup_{\epsilon\le t\le T}(|X(t)|/t^{1/2}) - b_T \le x\right) = \exp(-2e^{-x}), \tag{36}$$

where

$$a_T = (2\log\log T)^{1/2}$$
$$b_T = 2\log\log T + \frac{1}{2}\log\log\log T - \frac{1}{2}\log(4\pi).$$

For the proof of Theorem 4.1, we follow the arguments in J. Steinebach [14] and provide three lemmas.

Lemma 4.1. *For any $\epsilon > 0$, as $T \to \infty$*

$$a_T \sup_{\epsilon\le t\le r(T)}(|X(t)|/t^{1/2}) - b_T \to -\infty \ a.s. \tag{37}$$

and

$$a_T \sup_{\epsilon\le t\le r(T)}(|W(t)|/t^{1/2}) - b_T \to -\infty \ a.s., \tag{38}$$

where $r(T) = \exp(\log T)^p$ with some $0 < p < 1$.

Proof. From (21) of Section 3, we know that

$$|X(t) - W(t)| \overset{a.s.}{=} o(\{t/\log\log t\}^{1/2}) \text{ as } t \to \infty. \tag{39}$$

Hence, the law of iterated logarithm also holds for $\{X(t)\}$. This implies

$$\overline{\lim_{r\to\infty}}\,(2\log\log r)^{-1/2} \sup_{\epsilon\le t\le r}(|X(t)|/t^{1/2}) \le 1 \text{ a.s.}$$

Now, by our choice of $r(T)$, we have $\log\log r(T) = p\,\log\log T$, which, by $0 < p < 1$, completes the proof of (37), since $a_T(2\log\log r(T))^{1/2} - b_T \to -\infty$ as $T \to \infty$. Assertion (38) is proved by the same argument.

Lemma 4.2. *With $r(T)$ as in Lemma 4.1, we have as $T \to \infty$,*

$$a_T \sup_{r(T) \le t \le T} (|X(t) - W(t)|/t^{1/2}) \to 0 \ a.s. \tag{40}$$

Proof. From (39), as $r \to \infty$

$$\sup_{r \le t \le \infty} (|X(t) - W(t)|/t^{1/2}) \overset{a.s.}{=} o(\{\log\log r\}^{-1/2}).$$

The latter relation with $r = r(T) = \exp(\log T)^p$ immediately implies (40).

Lemma 4.3. *Let $\{W(t)\}_{t \ge 0}$ be a standard Wiener process. Then, for all real x, assertions (35) and (36) hold with $\epsilon = 1$ and $\{X(t)\}$ replaced by $\{W(t)\}$.*

Proof. By a simple transformation, the latter result is immediate from Theorem 1.9.1 of M. Csörgö and P. Révész [3] stated for the Ornstein-Uhlenbeck process (cf. also M.R. Leadbetter, G. Lindgren and H. Rootzén [9], Theorem 12.3.5).

Proof of Theorem 4.1. Combine Lemmas 4.1 – 4.3.

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