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On the Factorization of the Staircase Operators

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1. Introduction. Notations, definitions and main results

Let G, H be Hilbert spaces, dim $G \geq$ dim H. Consider the orthogonal sum $[+]_n \overline{G}_n \subset G$ and the orthogonal decomposition $H = [+]_n H_n$ where $\overline{G}_n \subset G$, and $H_n \subset H$, are closed linear spaces. Set:

$$\overline{\Lambda}_k = [+]_{n=1}^k \overline{G}_n, \quad L_k = [+]_{n=1}^k H_n, \quad k = 1, 2, \ldots .$$

Let $Z : H \to G$ be a linear bounded operator.

Definition 1. $Z = Z\{L_n, \overline{\Lambda}_n\}$ is called a staircase operator if there is a pair of sequences $\{L_n, \overline{\Lambda}_n\}$, $n = 1, 2, \ldots$ such that

$$(1) \qquad Z\,(L_n\,[-]\,L_{n-1}) \subset \overline{\Lambda}_n\,[-]\,\overline{\Lambda}_{n-2}, \quad n = 1, 2, \ldots,$$

where we put $L_0 = L_{-1} = \overline{G}_0 = \overline{G}_{-1} = 0$.

Definition 2. The pair of sequences $\{L_n\}$, $\{\overline{\Lambda}_n\}$ is said to has the chain property (resp. the staircase operator Z has the chain property) if

$$(2) \qquad \dim L_n \leq \dim \overline{\Lambda}_n \leq \dim L_{n+1}, \quad n = 1, 2, \ldots .$$

We consider operators $Z\{L_n, \overline{\Lambda}_n\}$ defined in a block form, i.e. represented by a matrix, whose rows and columns consist of operators called "blocks". The blocks of Z map (sums of) $H_i = L_i\,[-]\,L_{i-1}$ into (sums of) $\overline{G}_j = \overline{\Lambda}_j\,[-]\,\overline{\Lambda}_{j-1}$, $i, j = 1, 2, \ldots$. In consequence Z is defined via its restrictions on H_i and the restrictions of its range on $\overline{G}_j$. In this case the block-matrix notation

$$Z_{m-1,m}^{n-k,n} : L_n\,[-]\,L_{n-k} \to \overline{\Lambda}_m\,[-]\,\overline{\Lambda}_{m-1}, \quad n \geq k,\ m \geq 1$$

is used for a corresponding block of Z. The rows (resp. columns) of Z are indexed by a pair of lower (resp. upper) indeces thus fixing the endpoints of an index interval. If all blocks of Z, except possibly $Z^{n-1,n}_{n-1,n} \equiv Z_n$, are zero operators, then Z is called a block-diagonal operator, $Z = [+]\, Z_n$.

Let $H = [+]\, G_n$ be another orthogonal decomposition of H.

Denote $\Lambda_k = [+]^k_{n-1}\, G_n$ and require

$$\dim G_n = \dim \overline{G}_n; \tag{3}$$

$$\Lambda_{k-1} \subset L_k \subset \Lambda_k, \quad k = 1, 2, \dots, \quad \Lambda_0 = \{0\}. \tag{4}$$

Denote by $X = [+]\, X_n$ a block-diagonal operator such that $X_n : G_n \to \overline{G}_n$ and by $Y = [+]\, Y_n$ a block-diagonal operator such that $Y_n : H_n \to H_n$.

Definition 3. The pair of linear operators (X, Y), such that $Z = XY$, belongs to the factorization class F^a if X and Y are block-diagonal (as defined above) and bounded, Y is surjective and invertible with a bounded inverse Y^{-1}.

Let Z be a staircase operator with chain property. Our goal is to prove necessary and sufficient conditions for the factorization $Z = XY$. The conditions we propose permit an effective (explicit) construction of all possible pairs (X, Y) in the above factorization.

In R. Douglas's paper [1] conditions for factorization $C = AB$ are given in terms of operator ranges for general Hilbert space operators A, B, C (automatically satisfied in the present case). Additional information about the general theory can be found in M. Embry's paper [2].

The factorization presented here appeared originally in a particular case used to formulate a criterion on unitary implementability of some canonical transformations of Hamiltonians in infinite tensor products of Hilbert spaces in [3]. The factorization of an arbitrary staircase unitary Z into unitaries X and Y (with finite dimensional blocks, $\dim H_n < \infty$, $\dim G_n < \infty$, a conjecture of F. A. Berezin) was used in [3] and its proof published in [4].

Staircase operators with chain of finite dimensional subspaces (4) appear in the papers of D. A. Herrero [5], K. R. Davidson and D. A. Herrero [6], C. Foias, C. M. Pearcy and D. Voiculescu [7] in connection with bitriangular and biquasitriangular operators.

R. E. Curto and L. Fialkov [8] introduced a factorization technique to study quasisimilarity ($D \sim E$) of general Banach space operators D, E. This technique uses an effective factorization $C = AB$ of C taken from the commutant, e.g. of D and consequently our factorization procedure can be applied in this direction for special cases.

To state the theorem we need some more notation. Denote by

$$Z^t_n = Z^{n-1,n}_{n-2,n-1} : L_n\, [-]\, L_{n-1} \to \overline{\Lambda}_{n-1}\, [-]\, \overline{\Lambda}_{n-2},$$

$$Z^b_n = Z^{n-1,n}_{n-1,n} : L_n\, [-]\, L_{n-1} \to \overline{\Lambda}_n\, [-]\, \overline{\Lambda}_{n-1},$$

the "top" and "bottom" subblocks of Z_n (see Figure 1). Let R_n^t and R_n^b be closed linear subspaces of H with void intersection.

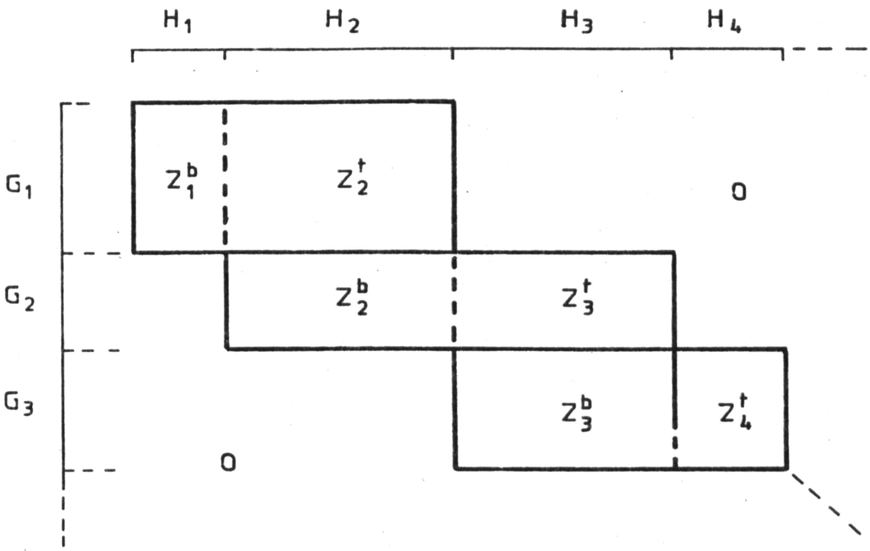

Fig. 1.

Consider the following conditions (5)–(7):

(5) $$R_n^t \subset Ker\, Z_n^t, \qquad R_n^b \subset Ker\, Z_n^b,$$

(6) $$\dim R_n^t = \dim(H_n \cap G_n), \qquad \dim R_n^b = \dim(H_n \cap G_{n-1}),$$

(7) $$R_n^t\,(+)\,R_n^b = H_n,$$

where "(+)" means direct, not necessarily orthogonal sum.

Theorem. *Let $Z \to Z\{L_n, \overline{\Lambda}_n\}$ be a staircase operator with chain property. Then the conditions (5)–(7) are necessary and sufficient for the factorization $Z = XY$ where $(X,Y) \in F^\alpha$.*

Proposition. *Every unitary staircase operator has the chain property.*

Corollary. *Let $Z\{L_n, \Lambda_n\}$ be an unitary staircase operator such that $\dim H_n < \infty$. Then Z is factorized, $Z = XY$, $(X,Y) \in F^\alpha$ and X,Y are unitaries.*

2. Proof of the factorization statements

To prove the factorization statements anounced in Sect. 1, we need some notations and remarks.

Set $H_n^l := H_n \cap G_{n-1}$, resp. $H_n^r := H_n \cap G_n$ for the "left", resp. "right" subspaces of H_n. It is straightforward to see that (4) implies

$$H_n = H_n^l \,[+]\, H_n^r, \tag{8}$$

$$L_n = \Lambda_{n-1} \,[+]\, H_n^r, \qquad \Lambda_{n-1} = L_{n-1} \,[+]\, H_n^l. \tag{9}$$

Consider a block-diagonal operator $K = [+]K_n$, $K_n = K_{n-1,n}^{n-1,n} : H_n \to H_n$.

In the sequel we use the same notation $K_n^{l(r)}$ for the extension by the zero operator of $K_n^{l(r)}$ from $H_n^{l(r)}$ to $H_n^{r(l)} [+] H_n^{l(r)}$, e. g. $K_n = K_n^l \,[+]\, K_n^r$, i. e. K_n is a sum of "left"(K_n^l) and "right" (K_n^r) subblocks of K_n.

Occasionally we use a block-matrix representation for Z, X and K with respect to the finer decomposition $[+]_n H = (H_n^l \,[+]\, H_n^r)$ which is a consequence of the decompositions $H = [+]_n H_n$ and $H = [+]_n G_n$ and the relations (4), (8). In these cases the matrix indeces of Z, K, X corresponding to the subspaces G_n are written with a bar — $\overline{n} = \overline{1}, \overline{2}, \ldots$ in order to distinguish them from the indeces corresponding to H_n, e. g.

$$K_n^l = K_{n-1,n}^{n-1,\overline{n-1}} : \Lambda_{n-1} [-] L_{n-1} \to L_n [-] L_{n-1},$$

$$K_n^r = K_{n-1,n}^{\overline{n-1},n-1} : L_n [-] \Lambda_{n-1} \to L_n [-] L_{n-1},$$

$n = 1, 2, \ldots$, $\overline{n} = \overline{1}, \overline{2}, \ldots$, where $\Lambda_0 = L_0 = \{0\}$.

Remark 1. In the sequel we use the following statements:

a) Suppose that $Ker\, K_n^l = Ker\, K_n^r = 0$. Then $Ker\, K_n = 0$ if and only if $Ran\, K_n^l \cap Ran\, K_n^r = \{0\}$.

b) Suppose that $Ker\, K_n = 0$ and take $g_n^{l(r)} \in Ran\, K_n^{l(r)}$, $g_n^l \,[+]\, g_n^r \equiv g_n$. Then $(K_n^l)^{-1} g_n = (K_n^l)^{-1} g_n^l \,[+]\, (K_n^r)^{-1} g_n^r$.

Indeed, to check a), let $Ker\, K_n^l = Ker\, K_n^r = \{0\}$. Assume on the contrary that $Ker\, K_n \neq 0$ and take $0 \neq f_n = f_n^l + f_n^r \in Ker\, K_n$, $f_n^{l(r)} \in H_n^{l(r)}$. We check that $f_n^{l(r)} \neq 0$. If e. g. $f_n^l = 0$ and $f_n^r \neq 0$ then $K_n f_n = K_n f_n^r = 0$ and $f_n^r \in Ker\, K_n^r$ contrary to the assumption. Hence

$$K_n^l f_n^l = -K_n^r f_n^r \neq 0,$$

i. e.

$$Ran\, K_n^l \cap Ran\, K_n^r \neq \{0\}.$$

Conversely, let $0 \neq g_n \in Ran\, K_n^l \cap Ran\, K_n^r$ and take $f_n^{l(r)} \in Dom\, K_n^{l(r)}$ and $g_n = K_n^{l(r)} f_n^{l(r)}$. Clearly, $0 \neq f_n^l - f_n^r \neq 0$.

Denote $f_n^- := f_n^l - f_n^r \neq 0$. Then $K_n f_n^- = K_n^l f_n - K_n^r = 0$ and hence $Ker\, K_n \neq \{0\}$.

b) is checked by direct multiplication of K_n with $(K_n^r)^{-1}\,[+]\,(k_n^l)^{-1}$.

Remark 2. We frequently use without reference the following assertion, which proof is straightforward. If the linear operator $A : H \ni Dom\, A \to Ran\, A \in G$ is bounded, surjective and invertible with bounded inverse, then $\dim \overline{(Dom\, A)} = \dim \overline{(Ran\, A)}$ ("—" stands for closure).

Proof of the theorem. Assume (5)–(7) to be valid. As a first step we construct a block-diagonal operator $K = [+]\, K_n$, $K_n : H_n \to H_n$ in the following way: Choose the orthogonal decomposition $H = [+]\, G_n$ so that G_n satisfies (3) and (4). Construct the operators:

$$\begin{array}{lcl} K_n^l & : H_n^l & \to R_n^b \\ K_n^r & : H_n^r & \to R_n^t \end{array} \tag{10}$$

to be bounded, invertible and surjective. Denote $\overline{K}_n = K_n^l\,[+]\,K_n^r$. Since $\overline{K}_n$ is bounded as a sum and is defined on H_n, then $\overline{K}_n$ has a closed graph. According to Remark 1, $\overline{K}_n$ is invertible due to $R_n^t \cap R_n^b = \{0\}$ (see (7)). Since $H_n = Ran\, \overline{K}_n = Dom\, \overline{K}_n^{-1}$ is closed and the graph of $\overline{K}_n^{-1}$ is closed, then $\overline{K}_n^{-1}$ is bounded as a consequence of the closed graph theorem. Let $||\overline{K}_n|| \leq \gamma_n$. Choose an arbitrary constant c and multiply $\overline{K}_n$ by $c_n = c/\gamma_n$. Denoting $c_n \overline{K}_n = K_n$ one has for $K = [+]\, K_n$ and $x = \sum_n x_n \in H$, $x_n \in H_n$:

$$||Kx||^2 = ||\sum_n K_n x_n||^2 \leq c \sum_n ||x_n||^2 = c\,||x||^2,$$

i. e. K is bounded. Since K^{-1} is invertible and surjective then K^{-1} is bounded.

Now we verify that the product $ZK =: X$ is a block-diagonal operator. Using blockwise multiplication one has:

$$X_{\overline{p-1,\overline{p}}}^{\overline{n-1,\overline{n}}} = \sum_\gamma Z_{\overline{p-1,\overline{p}}}^{\gamma}\, K_\gamma^{\overline{n-1,\overline{n}}}, \tag{11}$$

where the pair of summation indeces is denoted by a single index.

Note that conditions (5) and (10) can be combined to give for any n the following range conditions:

$$\text{a) } Ran\, K_{n+1}^l \subset Ker\, Z_{n+1}^b, \quad \text{b) } Ran\, K_n^r \subset Ker\, Z_n^t. \tag{12}$$

Relations (12) can be expressed as follows:

$$\text{a) } Z_{n+1}^b\, K_{n+1}^l = 0, \quad \text{b) } Z_n^t\, K_n^r = 0 \tag{13}$$

or equivalently as:

$$(14)\qquad \text{a)}\ Z^{n,n+1}_{\overline{\overline{n}},\overline{n+1}}\, K^{n,\overline{n}}_{n,n+1} = 0, \qquad \text{b)}\ Z^{n-1,n}_{\overline{n-2},\overline{n-1}}\, K^{\overline{n-1},n}_{n-1,n} = 0.$$

The staircase form of Z and the block-diagonal form of K together with the range conditions imply that X given by (11) is block-diagonal. Indeed, since $K_n : H_n \to H_n$ is block-diagonal and since $\{L_n . \Lambda_n\}$ satisfy (4), the inequality $K_{\gamma}^{\overline{n-1},\overline{n}} \neq 0$ is possible only if the index interval γ belongs to $(n-1, n+1)$. Due to the staircase condition (1) the only index intervals $\overline{\delta}$ for which $Z_{\overline{\delta}}^{n-1,n+1} \neq 0$ are those satisfying $\overline{\delta} \subset (\overline{n-2}, \overline{n+1}]$. Consequently for fixed upper index interval of X lying inside $(\overline{n-1}, \overline{n}]$, the only nonzero blocks of $X_{\overline{\delta}}^{\overline{n-1},\overline{n}}$ can be those, for which $\overline{\delta}$ belongs to one of the following three subintervals of $(\overline{n-2}, \overline{n+1}]$, namely: $\overline{\delta} \in (\overline{n-2}, \overline{n-1}]$, or $\overline{\delta} \in (\overline{n-1}, \overline{n}]$, or $\overline{\delta} \in (\overline{n}, \overline{n+1}]$.

We consider separately these three cases.

First note that the block $K_{n-1,n+1}^{\overline{n-1},\overline{n}}$ can be represented as the sum:

$$(15)\qquad K^{\overline{n-1},n}_{n-1,n}\,[+]\,K^{n,\overline{n}}_{n-1,n}\,[+]\,K^{\overline{n-1},n}_{n,n+1}\,[+]\,K^{n,\overline{n}}_{n,n+1},$$

where we used the same notation for the summands and their extensions by the zero operator to the whole space $L_n \setminus L_{n-1}$. The second and third members of (15) are zero operators since their upper and lower index intervals do not overlap (e. g. $(\overline{n-1}, n] \cap (n, n+1] = \phi$) and since K is block-diagonal.

For fixed $n = 1, 2, \ldots$, $\overline{n} = \overline{1}, \overline{2}, \ldots$ insert the first and last members of (15) in (11). Then the above three cases are

$$\text{(i)}\quad X^{\overline{n-1},\overline{n}}_{\overline{n-2},\overline{n-1}} = Z^{n-1,n}_{\overline{n-2},\overline{n-1}}\, K^{\overline{n-1},n}_{n-1,n} + Z^{n,n+1}_{\overline{n-2},\overline{n-1}}\, K^{n,\overline{n}}_{n,n+1}\,,$$

$$\text{(ii)}\quad X^{\overline{n-1},\overline{n}}_{\overline{n-1},\overline{n}} = Z^{n-1,n}_{\overline{n-1},\overline{n}}\, K^{\overline{n-1},n}_{n-1,n} + Z^{n,n+1}_{\overline{n-1},\overline{n}}\, K^{n,\overline{n}}_{n,n+1}\,,$$

$$\text{(iii)}\quad X^{\overline{n-1},\overline{n}}_{\overline{\overline{n}},\overline{n+1}} = Z^{n-1,n}_{\overline{\overline{n}},\overline{n+1}}\, K^{\overline{n-1},n}_{n-1,n} + Z^{n,n+1}_{\overline{\overline{n}},\overline{n+1}}\, K^{n,\overline{n}}_{n,n+1}\,.$$

The first product at the right-hand side of (i) and the second product at the right-hand side of (iii) are zero due to the range conditions (12) b) and (12) a), resp. The second product at the right-hand side of (i) and the first product at the right-hand side of (iii) are zero due to the staircase condition (1) for Z. Consequently, X is block-diagonal and the n-th block X_n of X can be written with the aid of the notations $Z_n^{t(b)}$ and $K_n^{l(r)}$ as in the case (ii) as follows:

$$(16)\qquad X_n = Z_n^b\, K_n^r + Z_{n+1}^t\, K_{n+1}^l, \quad n = 1, 2, \ldots .$$

Denote $Y_n := K_n^{-1}$ and $Y := [+]Y_n$. Then $Y = K^{-1}$ and $ZK = X$ is bounded, since Z and K are bounded. In consequence $(X, Y) \in F^{\alpha}$.

Conversely, assume $Z = Z\{L_n, \overline{\Lambda}_n\}$ be a staircase operator with chain property. Suppose $Z = XY$ with $(X,Y) \in F^\alpha$. We verify the conditions (5)–(7). Denote $K_n = Y_n^{-1} : Ran\, Y \to H_n$. The assumptions imply that $Ran\, Y_n = H_n$, K_n is bounded (with bounded inverse) and K_n is surjective and invertible. Consequently the restrictions $K_n^{l(r)}$ of K_n on $H_n \cap G_{n-1}$ (resp. on $H_n \cap G_n$), namely:

$$\begin{aligned} K_n^l &: H_n \cap G_{n-1} \to H_n \\ K_n^r &: H_n \cap G_n \to H_n \end{aligned} \tag{17}$$

are invertible. Then Remark 1 a) implies:

$$Ran\, K_n^l \cap Ran\, K_n^r = \{0\} \tag{18}$$

and since K_n^{-1} is bounded then $(K_n^{l(r)})^{-1}$ are bounded. Furthermore, $K_n^{l(r)}$ have closed graphs because they are defined on closed spaces and are bounded by assumption. In consequence, $K_n^{l(r)}$ are invertible, have closed graphs and bounded inverses. One can prove (see e. g. Plessner [9], ch. 6, Theorem 6.2.2) that this is equivalent to $K_n^{l(r)}$ to have close ranges.

Set $R_n^{b(t)} := Ran\, K_n^{l(r)}$. Since K_n is surjective, then (18) and the decomposition (8) imply

$$Ran\, K_n^l \,(+)\, Ran\, K_n^r = Ran\, K_n = H_n,$$

i. e. (7) is verified.

Furthermore, the staircase form of Z and the block-diagonal forms of X and Y imply via the equality $Z = XY$ the range conditions (12) a),b). In more detail, as done in the direct part of the proof, one starts from the equality $X = ZK$ (i. e. (11)) and considers the three cases (i)–(iii). Then one concludes again that (11) has the form (16) only by comparing the left-hand side and right-hand side of (11) and using that X is block-diagonal. We are interested here in the zero summands $Z_\alpha^\gamma K_\gamma^\beta$ discussed above in the cases (i) and (iii). As already was indicated there, two of these summands (from the right-hand sides of (i) and (iii) correspondingly) are zero operators because of the staircase condition of Z. The annulation of each of the remaining at the right-hand sides of (i) and (iii) two summands expresses exactly the range conditions (12).

With the notation $R_n^{t(b)} := Ran\, K_n^{l(r)}$ used above, (12) is the same as (5).

Since $K_n^{l(r)}$ are invertible and surjective, Remark 2 implies:

$$\begin{aligned} \dim\, Ran\, K_n^l &= \dim(H_n \cap G_{n-1}) \\ \dim\, Ran\, K_n^r &= \dim(H_n \cap G_n) \end{aligned} \tag{19}$$

Since by assumption $Ran\,Y = H_n$, using the notation $Ran\,K_n^{l(r)} = R_n^{t(b)}$ we conclude that (6) is valid being identical with (19). The proof is completed.

Remark 3. The proof of the theorem gives for a fixed Z an effective factorization procedure for explicit construction of all the pairss $(X,Y) \in F^\alpha$ via the operators $K_n^{l(r)}$ defined in (10).

Proof of the proposition. The chain property (2) implies

$$ZL_k \subset \overline{\Lambda}_k, \qquad Z^{-1}\overline{\Lambda}_k \subset L_{k+1}. \tag{20}$$

Since Z is unitary, (20) and Remark 1 imply:

$$\dim L_k = \dim ZL_k \leq \dim \overline{\Lambda}_k, \qquad \dim L_k = \dim Z^{-1}\overline{\Lambda}_k \leq \dim L_{k+1}.$$

These inequalities express the chain property.

Proof of the corollary. Denote by $Z^{(n)}$, $Z^{(n,t)}$ the following blocks of Z (see also Figure 1):

$$Z^{(n)} = Z : L_n \to \Lambda_n, \qquad Z^{(n,t)} = Z : L_{n+1} \to \Lambda_n .$$

Note that since Z is unitary and of staircase type, then the orthogonality of different rows (columns) of Z is reduced to orthogonality of the rows Z_n^t to the rows of Z_n^b (resp. the columns of Z_{n+1}^t to the columns of Z_n^b). In consequence it is straightforward to check using (8), (9) the following relations:

$$\begin{aligned}
\dim \Lambda_n &= rank\, Z^{(n,t)},\\
\dim L_n &= rank\, Z^{(n)},\\
rank\, Z_{n+1}^t &= rank\, Z^{(n-1,t)} - rank\, Z^{(n-1)} = \dim \Lambda_{n-1} - \dim L_{n-1} = \dim H_n^l,\\
rank\, Z_n^b &= rank\, Z^{(n)} - rank\, Z^{(n-1,t)} = \dim L_n - \dim \Lambda_{n-1} = \dim H_n^r.
\end{aligned}$$

With the aid of the general relation

$$\dim Ker\, Z_n^{t(b)} = \dim H_n\,[-]\,rank\, Z_n^{t(b)}$$

and (8) one verifies (5), where $R_n^{t(b)} = Ker\, Z_n^{t(b)}$. At the same time we verified (7) as a consequence of (8) and of $H_n = Ker\, Z_n^t\,[+]\,Ker\, Z_n^b$. (The latter equality is due to the orthogonality of the rows of Z_n^t to the rows of Z_n^b).

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References

[1] R. G. Douglas. On majorization, factorization and range inclusion of operators on Hilbert spaces. *Proc. Amer. Math. Soc.*, **17**, 1966.

[2] M. Embry. Factorization of operators in Hilbert space. *Proc. Amer. Math. Soc.*, **38**, 1973, 507–590.

[3] V. N. Sushko, T. S. Todorov. Unitary implementability of a type of canonical transformations. *Compt. Rend. Acad. Bulg. Sci.*, **22**, 1969, 1349–1351 (in Russian).

[4] T. S. Todorov. On the factorization of a type of infinite unitary matrices. *Annuaire des Ecoles Tech. Superiores – Phys. (Sofia)*, **9**, 1975, 107–116.

[5] D. A. Herrero. All (all?) about triangular operators. Preprint, 1988.

[6] K. A. Davidson, D. A. Herrero. The Jordan form of a bitriangular operator. *J. Funct. Anal.*.

[7] C. Foias, C. M. Pearcy, D. Voiculescu. The staircase representation of a biquasitriangular operator. *Michigan Matn. J.*, **22**, 1975, 343–352.

[8] R. E. Curto, L. A. Fialkov. Similarity, quasisimilarity and operator factorizations. *Trans. Amer. Matn. Soc.*, **314**, 1989, 225–254.

[9] A. I. Plesner. Spectral Theory of Linear Operators. Nauka, Moscow, 1965 (in Russian).

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