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Hölder and Minkowski Inequalities in Hilbert Space

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1. Introduction

Let A be a self-adjoint operator on a Hilbert space H. We say that

$$mI \leq A \leq MI,$$

where I is the identity operator, if

$$m(x,x) \leq (Ax,x) \leq M(x,x)$$

for all $x \in H$.

The following three theorems were proved by Mond and Shisha in [1]:

Theorem A. *Let B, C be permutable, bounded, self-adjoint operators, having positive lower bounds. Let $p>1$, $p^{-1}+q^{-1}=1$, $0<\alpha<\beta$, and suppose*

$$\alpha I \leq B^{1/q}C^{-1/p} \leq \beta I. \tag{1}$$

Then, for every point $z(\neq 0)$ of H,

$$1 \leq \frac{(B^p z,z)^{1/p}(C^q z,z)^{1/q}}{(Bz,Cz)} \leq K \tag{2}$$

where

$$K = \left[\frac{q}{p+q}\frac{\gamma^p-\gamma^{-q}}{1-\gamma^{-1}}\right]^{1/q}\left[\frac{p}{p+q}\frac{\gamma^p-\gamma^{-q}}{\gamma^p-1}\right]^{1/q} \tag{3}$$

and $\gamma = \beta/\alpha$.

Theorem B. *Assume the hypotheses of Theorem A, and set*

$$h(t) \equiv t^{1/p} - (at+b)^{-1/q}$$

$$a = (\beta^{-q} - \alpha^{-q})/(\beta^p - \alpha^p), \quad b = (\beta^p\alpha^{-q} - \beta^{-q}\alpha^p)/(\beta^p - \alpha^p). \tag{4}$$

Then for every point $z(\neq 0)$ *of* H,

$$0 \le [(B^p z, z)/(Bz, Cz)]^{1/p} - [(Bz, Cz)/(C^q z, z)]^{1/q} \le \tilde{\Delta} \tag{5}$$

where $\tilde{\Delta} = \max_{\alpha^p \le t \le \beta^p} h(t) = h(t^*)$, t^* *being the unique solution of* $h'(t) = 0$ *in* (α^p, β^p).

Theorem C. *Assume the hypotheses of Theorem A, except that instead of (1), assume*

$$\alpha I \le [B(B+C)^{-1}] \le \beta I, \quad \alpha I \le [C(B+C)^{-1}]^{1/q} \le \beta I. \tag{6}$$

Then with $\gamma = \beta/\alpha$, *we have for every point* $z \neq 0$ *of* H,

$$1 \le \frac{(B^p z, z)^{1/p} + (C^p z, z)^{1/p}}{((B+C)^p z, z)^{1/p}} \le K \tag{7}$$

where K *is defined by (3).*

Moreover, with the same proofs as in [1], we can give extensions of the previous results. Namely, let $q < 0$, so that $0 < p < 1$ and $p^{-1} + q^{-1} = 1$. Then the reverse inequalities in (2) and (7) hold.

In this paper, we give extensions of the previous results to the case of several operators.

Results

Theorem 1. *Let* p, q, α, β *be real numbers such* $p^{-1} + q^{-1} = 1$, $0 < \alpha < \beta$. *Further, let* B_k, C_k *be permutable, bounded, self-adjoint operators having positive lower bounds such that*

$$\alpha I \le B_k^{1/q} C_k^{-1/p} \le \beta I \tag{8}$$

for every $k = 1, 2, \ldots, n$. *If* $z_k \in H$ $(z_k \neq 0)$, *then if* $p > 1$,

$$1 \le \frac{\left[\sum_{k=1}^n (B_k^p z_k, z_k)\right]^{1/p} \left[\sum_{k=1}^n (C_k^q z_k, z_k)\right]^{1/q}}{\sum_{k=1}^n (B_k z_k, C_k z_k)} \le K \tag{9}$$

where K is defined by (3). If $0 < p < 1$, the reverse inequality holds in (9).

P r o o f. This first inequality in (9) is a consequence of the first inequality in (2) and the classical Hölder inequality for sums. Indeed, we have, in the case $p > 1$,

$$\left[\sum_{k=1}^{n}(B_k^p z_k, z_k)\right]^{1/p}\left[\sum_{k=1}^{n}(C_k^q z_k, z_k)\right]^{1/q} \geq$$

$$\geq \sum_{k=1}^{n}(B_k^p z_k, z_k)^{1/p}(C_k^q z_k, z_k)^{1/q} \geq \sum_{k=1}^{n}(B_k z_k, C_k z_k).$$

For $p < 1$, we have the reverse inequality.

On the other hand, the following operator inequality is given in [1]:

$$r(A^r - uA^s - vI) \geq 0 \tag{10}$$

if $s > r$, A is a positive operator with

$$\alpha I \leq A \leq \beta I,$$

$$u = (\beta^r - \alpha^r)/(\beta^s - \alpha^s), \quad v = (\beta^s \alpha^r - \beta^r \alpha^s)/(\beta^s - \alpha^s).$$

Setting $s = p$, $r = -q$, $A = B_k^{1/q} C_k^{-1/p}$, we get

$$qC_k^q - aqB_k^p \leq bqB_k C_k \tag{11}$$

where a and b are defined as in (4).

Thus for every $z_k \in H$,

$$q(C_k^q z_k, z_k) - aq(B_k^p z_k, z_k) \leq bq(B_k z_k, C_k z_k).$$

Summation gives

$$q\sum_{k=1}^{n}(C_k^q z_k, z_k) - aq\sum_{k=1}^{n}(B_k^p z_k, z_k) \leq bq\sum_{k=1}^{n}(B_k z_k, C_k z_k). \tag{12}$$

Suppose $q > 1$, $p > 1$. Then (12) is equivqlent to

$$\frac{1}{q}\left[q\sum_{k=1}^{n}(C_k^q z_k, z_k)\right] + \frac{1}{p}\left[-aq\sum_{k=1}^{n}(B_k^p z_k, z_k)\right] \leq b\sum_{k=1}^{n}(B_k z_k, C_k z_k). \tag{13}$$

Using the AG (Arithmetic–Geometric) inequality, we obtain

$$\left[q\sum_{k=1}^{n}(C_k^q z_k, z_k)\right]^{1/q}\left[-aq\sum_{k=1}^{n}(B_k^p z_k, z_k)\right]^{1/p} \leq b\sum_{k=1}^{n}(B_k z_k, C_k z_k) \tag{14}$$

which is second inequality in (9). If $q < 0$, the reverse inequality in (13) holds and, in this case, we use the reverse AG inequality.

Theorem 2. *Assume the hypotheses of Theorem 1 and let a, b, be defined by (4). If* $z_k(\neq 0)$ *are points from H, then, for* $p > 1$,

$$(15)\qquad 0 \leq \left\{ \frac{\sum_{k=1}^n (B_k^p z_k, z_k)}{\sum_{k=1}^n (B_k z_k, C_k z_k)} \right\}^{1/p} - \left\{ \frac{\sum_{k=1}^n (B_k z_k, C_k z_k)}{\sum_{k=1}^n (C_k^q z_k, z_k)} \right\}^{1/q} \leq \tilde{\Delta}$$

where $\tilde{\Delta}$ *is defined as in Theorem B.*

Proof. (12) is equivalent to

$$\frac{\sum_{k=1}^n (C_k^q z_k, z_k)}{\sum_{k=1}^n (B_k z_k, C_k z_k)} - a \frac{\sum_{k=1}^n (B_k^p z_k, z_k)}{\sum_{k=1}^n (B_k z_k, C_k z_k)} \leq b.$$

Thus

$$\left\{ \frac{\sum_{k=1}^n (B_k^p z_k, z_k)}{\sum_{k=1}^n (B_k z_k, C_k z_k)} \right\}^{1/p} - \left\{ \frac{\sum_{k=1}^n (B_k z_k, C_k z_k)}{\sum_{k=1}^n (C_k^q z_k, z_k)} \right\}^{1/q} \leq$$

$$\leq \left\{ \frac{\sum_{k=1}^n (B_k^p z_k, z_k)}{\sum_{k=1}^n (B_k z_k, C_k z_k)} \right\}^{1/p} - \left\{ a \frac{\sum_{k=1}^n (B_k^p z_k, z_k)}{\sum_{k=1}^n (B_k z_k, C_k z_k)} + b \right\}^{-1/q} =$$

$$= h \left\{ \frac{\sum_{k=1}^n (B_k^p z_k, z_k)}{\sum_{k=1}^n (B_k z_k, C_k z_k)} \right\} \leq \max_{t \in (\alpha^p, \beta^p)} h(t) = h(t^*).$$

Thus we have proved the second inequality in (15). moreover, the first inequality in (15) is, in fact, the first inequality in (9).

Theorem 3. *Assume the hypotheses of Theorem 1, except that instead of (8), assume*

$$(16)\qquad \alpha I \leq [B_k(B_k + C_k)^{-1}]^{1/q} \leq \beta I, \quad \alpha I \leq [C_k(B_k + C_k)^{-1}]^{1/q} \leq \beta I$$

for every $k = 1, 2, \ldots, n$. *If* $p > 1$ *and* $z_k \neq 0$ $(k = 1, \ldots, n)$ *are points from H. Then*

$$(17)\qquad 1 \leq \frac{\left[\sum_{k=1}^n (B_k^p z_k, z_k)\right]^{1/p} + \left[\sum_{k=1}^n (C_k^p z_k, z_k)\right]^{1/p}}{\left[\sum_{k=1}^n ((B_k + C_k)^p z_k, z_k)\right]^{1/p}} \leq K$$

where K is given by (3). If $p < 0$, *the inequality in (17) holds.*

Proof. The inequality in (17) is a simple consequence of the first inequality in (7) and the classical Minkowski inequality for sums. If $p > 1$, then

$$\left[\sum_{k=1}^n (B_k^p z_k, z_k)\right]^{1/p} + \left[\sum_{k=1}^n (C_k^p z_k, z_k)\right]^{1/p}$$

$$\geq \left[\sum_{k=1}^n \{(B_k^p z_k, z_k)^{1/p} + (C_k^p z_k, z_k)^{1/p}\}^p\right]^{1/p}$$

$$\geq \left[\sum_{k=1}^n ((B_k + C_k)^p z_k, z_k)\right]^{1/p}.$$

For $p < 1$, we have the reverse inequalities.

The second inequality is a consequence of Theorem 1. For $p > 1$,

$$\begin{aligned}\sum_{k=1}^{n}((B_k + C_k)^p z_k, z_k) &= \sum_{k=1}^{n}(B_k(B_k + C_k)^{p-1} z_k, z_k) \\ &\quad + \sum_{k=1}^{n}(C_k(B_k + C_k)^{p-1} z_k, z_k) \\ &\geq K^{-1}\left\{\left[\sum_{k=1}^{n}(B_k^p z_k, z_k)\right]^{1/p}\left[\sum_{k=1}^{n}((B_k + C_k)^p z_k, z_k)\right]^{1/q}\right. \\ &\quad \left. + \left[\sum_{k=1}^{n}(C_k^p z_k, z_k)\right]^{1/p}\left[\sum_{k=1}^{n}((B_k + C_k)^p z_k, z_k)\right]^{1/q}\right\}.\end{aligned}$$

This is equivalent to the second inequality in (17). For $p < 1$, the reverse inequalities hold.

Reference

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