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Some Properties of Nonlinear Degenerate Parabolic Equations[1]

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1. Introduction

The purpose of this paper is to study non-negative continuous weak solutions of the equation

$$(1) \qquad u_t = \operatorname{div}(u.V)$$

in the strip $S_T = (0,T) \times \mathbb{R}^d$, $d \geq 1$. Vector function V has the form

$$V = \left(\psi^1(\nabla\Gamma(u)), \dots, \psi^d(\nabla\Gamma(u))\right),$$

where $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$, $\psi^j(q) = \frac{\partial}{\partial q_j}\psi(q)$, $j = 1, 2, \dots, d$, $\psi(q)$ is a smooth, non-negative, convex function, homogeneous of degree $p > 1$ and $\Gamma(s)$ is continuously differentable and such that $s\Gamma'(s) > 0$ for $s \neq 0$.

Our main assumption is a regularizing effect for the solutions u of (1) in the form:

$$(2) \qquad \left| \begin{array}{l} \textit{There exists a non-negative function } E(s) \textit{ such that} \\ \operatorname{div} V \geq -\dfrac{E(u)}{t} \end{array} \right.$$

in the sense of $\mathcal{D}'(\mathbb{R}^d)$.

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Under such a condition we obtain in section 2 a pointwise estimate for the solutions of (1). In section 3 we study some properties of such solutions as an estimate on the support of the solutions with compact initial data, $L^\infty - L^1$ estimate and an estimate on the initial trace. In section 4 we give some sufficient conditions for obtaining (2) and some comments.

Important examples of (1) are the following equations:

$$u_t = \Delta\varphi(u), \tag{3}$$

i.e. (1) with $\psi(q) = \frac{1}{2}\sum_1^d q_j^2$, $\varphi'(s) = s\Gamma'(s)$;

$$u_t = \sum_1^d \frac{\partial}{\partial x_j}\left(|\frac{\partial}{\partial x_j}a(u)|^{p-2}\frac{\partial}{\partial x_j}a(u)\right), \tag{4}$$

often with $a(u) = |u|^{m-1}u$, where $\psi(q) = \frac{1}{p}\sum_1^d |q_j|^p$, $a'(s) = |s|^{p'-2}\, s\,\Gamma'(s)$, $p' = \frac{p}{p-1}$, $m>0$;

$$u_t = \sum_1^d \frac{\partial}{\partial x_j}\left(|\nabla a(u)|^{p-2}\frac{\partial}{\partial x_j}a(u)\right) \tag{5}$$

with $\psi(q) = \frac{1}{p}\sum_1^d |q_j|^p$ and with the same $a(s)$ as in (4).

These equations were studied from various points of view with naturally different methods — see recent survey in A. Kalashnikov [1].

As an illustration of our hypothesis (2) and further results, we shall write down here two families of G. Barenblatt type solutions [11] of (1) in the case $\Gamma_n = \frac{1}{n}|s|^n$, $n>0$ and $\Gamma_0 = \ln|s|$:

$$B_{i,n}(t,x) = n^{1/n}\left(\left\{c_{i,n}|b_{i,n}(t)|^{-\alpha} + b_{i,n}(t)\;\theta\left(\frac{x-x_i}{b_{i,n}(t)}\right)\right\}_+\right)^{1/n},\ n>0; \tag{6}$$

$$B_{i,0}(t,x) = c_{i,0}|b_{i,0}(t)|^{-d/p}\exp\left(b_{i,0}(t)\,\theta(\frac{x-x_i}{b_{i,0}(t)})\right),\ n=0, \tag{7}$$

where $i=1,2$, $\{s\}_+ = \max(s,0)$, $c_{i,n}, c_{i,0}$ – positive constants, $\theta(s)$ is Joung connugate to $\psi(q)$, $\alpha = \frac{dn}{\varkappa}$, $\varkappa = dn(p-1)+p$, with $b_{1,n}(t) = -\varkappa(t+\tau)$, $\tau \geq 0$, this is compactly supported solution "fundamental", if $\tau = 0$ (the initial trace is δ-function), defined in $(0,\infty)\times \mathbb{R}^d$, with $b_{2,n}(t) = \varkappa(T-t)$

— this is a "blow-up" solution, defined on $(0,T)\times \mathbb{R}^d$, $b_{1,0}(t) = -p(t+\tau)$, $b_{2,0}(t) = p(T-t)$. We can see that on the support of $B_{i,n}$ the components of the velocities are

$$V^j_{i,n} = \frac{x_j - x_{ij}}{b_{i,n}(t)}, \quad i = 1,2, \quad j = 1,\ldots,d$$

and

$$\operatorname{div} V_{i,n} = \frac{d}{b_{i,n}(t)} \geq -\frac{d}{\varkappa t}$$

with equality for the "fundamental" case $n>0$, $i=1$, $\tau=0$.

Note that a part of our results has been announced in [16].

2. Pointwise estimate

For many purposes a very good instrument is a pointwise estimate — an analogue of the classical J. Mozer inequality [2] between two values of the solution.

Theorem 1 (pointwise estimate). *Let $u(t,x)$ be a continuous, nonnegative weak solution of (1) in S_T for which (2) is fulfilled and*

(i) If there exists a constant α such that $E(u) \leq \frac{\alpha\Gamma(u)}{u\Gamma'(u)}$, then

$$(8) \qquad t^{\alpha}\Gamma(u(t,x)) \leq s^{\alpha}\Gamma(u(s,y)) + p\,\theta\Big(\frac{y-x}{p}\Big)\Big[\frac{s^{\sigma}-t^{\sigma}}{\sigma}\Big]^{-\frac{1}{p-1}}$$

for $0<t<s<T$, where $\sigma = 1-(p-1)\alpha$;

(ii) If there exists a constant λ such that $E(u) \leq \frac{\lambda}{u\Gamma'(u)}$, then

$$(9) \qquad \Gamma(u(t,x)) \leq \Gamma(u(s,y)) + \lambda \ln\frac{s}{t} + p\,\theta\Big(\frac{y-x}{p}\Big)[s-t]^{-\frac{1}{p-1}}$$

for $0<t<s<T$.

Proof. Denote $g(s) = \gamma(s)/\gamma'(s)$, where $\gamma(s) = \Gamma^{-1}(s)$. Note that since $s\Gamma'(s) > 0$ it follows that $g(s) > 0$ for $s \neq 0$. From (1) there follows that the function $v = \Gamma(u)$ satisfies the equation

$$(10) \qquad v_t = g(v)\operatorname{div} V + p\,\psi(\nabla v).$$

From (11), (i) and (10) the next differential inequality takes place for v:

$$(11) \qquad v_t \;\geq\; -\frac{\alpha}{t}\, v + p\,\psi(\nabla v).$$

Define the function

$$(12) \qquad h(t) \;=\; t^\alpha v(t, x(t)) + p\int_{t_0}^{t} s^\alpha\; \theta(\frac{1}{p}\;\dot{x}(s))\, ds,$$

where $x(s)$ is differentiable vector-function and function

$$\theta(s) = \sup_q[\sum_1^d q_j s_j - \psi(q)]$$

is the connugate to the function $\psi(q)$. Using (12) and the Joung inequality for convex adjoint functions ψ and θ, we obtain

$$\begin{aligned} h'(t) \;&=\; \alpha t^{\alpha-1} v \;+\; t^\alpha v_t \;+\; t^\alpha \;\; sum_1^d\, v_{x_j}\, \dot{x}_j \;+\; p t^\alpha\; \theta(\frac{1}{p}\;\dot{x}(t)) \\ &\geq\; p t^\alpha \left[\psi(\nabla v) \;+\; \theta(\frac{1}{p}\;\dot{x}(t)) \;+\; \sum_1^d v_{x_j}\,\frac{\dot{x}_j}{p}\right] \;\geq\; 0. \end{aligned}$$

Then for $0<t<s<T$

$$(13) \qquad t^\alpha\; v(t, x(t)) \;\leq\; s^\alpha\; v(s, x(s)) \;+\; p\int_t^s \tau^\alpha\; \theta(\frac{1}{p}\;\dot{x}(\tau))\, d\tau.$$

To obtain (8) take the vector

$$(14) \qquad \overline{x}(\tau) \;=\; x + (y - x)\,\beta(\tau), \quad t \leq \tau \leq s$$

with $\beta(t) = 0$, $\beta(s) = 1$ and such that $\overline{x}(\tau)$ minimizes the functional

$$(15) \qquad \int_t^s \tau^\alpha\; \theta(\frac{1}{p}\;\dot{x}(\tau))\, d\tau \;=\; \theta(\frac{y-x}{p}) \int_t^s \tau\alpha\; \dot{\beta}^{p'}(\tau)\, d\tau$$

on the set of functions $\{x(\tau),\;\; x(t) = x,\;\; x(s) = y\}$.

From the Hölder inequality we get

$$1 = \int_t^s \beta(\tau)\,d\tau = \int_t^s \tau^{\alpha/p'}\,\dot\beta(\tau)\,\tau^{-\alpha/p'}\,d\tau \le$$
$$\le \left(\int_t^s \tau\alpha\,\dot\beta^{p'}(\tau)\,d\tau\right)^{\frac{1}{p'}} \left(\int_t^s \tau^{-\alpha(1-p)}\,d\tau\right)^{1/p}.$$

Then

$$\int_t^s \tau^{\alpha}\,\dot\beta^{p'}(\tau)\,d\tau \ \ge\ \left(\frac{s^\sigma - t^\sigma}{\sigma}\right)^{-\frac{1}{p-1}}$$

and the equality takes place for

$$\beta(\tau) = \frac{\tau^\sigma - t^\sigma}{s^\sigma - t^\sigma}, \quad \sigma = 1-(p-1)\alpha. \tag{16}$$

To prove the case (ii) we get the function

$$h(t) = \lambda \ln t + v(t,x(t)) + p\int_{t_0}^t \theta\left(\frac{\dot x(\tau)}{p}\right)d\tau \tag{17}$$

and the same way as for (9).

Remark 1. Note that the above calculations are formal for the solutions of (1) which are not smooth, but under suitable regularization and limited process as in [15] we obtain such a pointwise estimate for nonnegative continuous weak solutions of (1). Such a pointwise estimates (8) or (9) are an analogue of the classical J. Mozer [2] inequality between two values of the solution. In [2] such an estimate as (9) was proved for a very large class of linear equations — with measurable coefficients, using the Harnak inequality for such equations. In the nonlinear case, for example for equation (3) with $s\Gamma'(s) \to 0$ for $s \to 0$, due to the compactness of support of the solutions with compact initial data, the Harnak inequality fails, however the pointwise estimate (8) takes place under some additional conditions on φ as was proved in [8].

Remark 2. In some sense acording to $E(u)$, (8) and (9) are equivalent to the regularizing effect (2). Indeed, since the vector function (14) with $\beta(s)$ in the form (16) minimizes the corresponding functional (14), then from (8) there follows (13) for arbitrary $x(\tau)$. Now let $x(\tau)$ be a solution of the Cauchy problem

$$\begin{cases} \frac{1}{p}\,\dot x_j(\tau) = -\psi_j(\nabla v(\tau, x(\tau))), & t \le \tau \\ x_j(t) = x_j, & j = 1,2,\ldots,d. \end{cases} \tag{18}$$

Then since

$$\psi(\nabla v) + \theta\left(\frac{\dot x}{p}\right) + \frac{1}{p}\sum_1^d v_{x_j}\dot x_j = 0$$

it follows that $h(\tau)$ is monotonous and $g(v)\,\mathrm{div}\,V \ge -\frac{\alpha}{t}\,v$.

3. Applications to the Cauchy problem

Using the pointwise estimate it is not difficult to obtain some properties for the solutions of Cauchy problem for the equation (1).

Theorem 2 ($L^\infty - L^1$ estimate). *Let u be a weak solution of (1) in S_T, which satisfies the initial condition $u(0,x) = u_0(x) \in L^1(\mathbb{R}^d)$ in the sense of $\mathcal{D}'(\mathbb{R}^d)$. Under the conditions of Theorem 1 and $\sigma > 0$, there exists a nonnegative function $K(s)$ such that $K(s) = O(((1-\sigma)\Gamma(s))^\sigma)$ for small s and*

$$K(s) = O(s^{p\alpha/d})$$

for large s, and

$$\Gamma(u(t,x)) \le \frac{1}{1-\sigma} K(||u_0||_{L^1}\ t^{-d/p}). \tag{19}$$

Proof. Integrating (8) and (9) over the ball

$$B_R^\theta(x) = \{y : p\,\theta(\frac{y-x}{p}) \le R^{p'}\}$$

for $s = \delta t$, $\delta > 1$ and applying the conservation mass law as in [8] for the equation (1), we get for any $\delta > 1$

$$\Gamma(u(t,x)) \le (\frac{\delta^\sigma - 1}{\sigma})^{-\frac{1}{p-1}}\ t^{-\frac{1}{p-1}}\ R^{p'} + \delta^\alpha\ \Gamma(\frac{||u_0||_{L^1}}{\omega R^d}), \tag{20}$$

$$\Gamma(u(t,x)) \le \Gamma(\frac{||u_0||_{L^1}}{\omega R^d}) + \beta \ln\delta + (\delta - 1)^{-\frac{1}{p-1}}\ R^{p'}\, t^{-\frac{1}{p-1}}, \tag{21}$$

where ω is volume of B_1^θ. Since $\sigma > 0$ then

$$\frac{\delta^\sigma - 1}{\sigma} \ge \min(\frac{1}{\sigma}, 1)\ \delta^{\sigma-1}(\delta - 1) \ge \min(\frac{1}{\sigma}, 1)\ \frac{\epsilon}{1+\epsilon}\ \delta^\sigma$$

for $\delta > 1 + \epsilon$ and we can minimize the right side in $\delta > 1 + \epsilon$.

Remark 3. for $\Gamma_n(s) = \frac{1}{n}|s|^n$ and $\Gamma_0(s) = \ln|s|$ it is easy to minimize (20) and (21) over $R > 0$ and then over $\delta > 1$:

$$u(t,x) \le \left[n(\frac{d}{p'})^\sigma\ \frac{\sigma^\alpha}{1-\sigma}\right]^{1/n}\ t^{-\alpha/n}(\frac{||u_0||_{L^1}}{\omega})^\sigma, \quad n > 0, \tag{22}$$

(23) $$u(t,x) \;\leq\; \frac{||u_0||_{L^1}}{\omega}\, t^{-d/p}(\frac{d}{p'})^{d/p'}\, \exp(\frac{d}{p'}), \quad n=0.$$

For the equation (4) with $a(u) = |u|^{m-1}u$ such an estimate (22) was obtained in [14] for $d = 1$, and for the equation (3) a similar estimate as (19) under different conditions on φ was obtained in many works — see references in [1]. We mention that for a very large class of φ the estimate like (19) was obtained in [6].

In some cases when $s\Gamma'(s) \to 0$ for $s \to 0$ we obtain finite speed of propagation.

Theorem 3 (compactness of the support). *Let* $\Gamma(s) = \frac{1}{n}|s|^n$, $n>0$ *and the assumptions of Theorem 2 be fulfilled. If* $\operatorname{supp} u_0 \subset B^\theta_{R_0}(x_0)$ *then* $\operatorname{supp} u(t,\cdot) \subset B^\theta_{R(t)}(x_0)$, *where*

(24) $$R(t) \;\leq\; R_0 + ct^{\sigma/p}[\Gamma(||u_0||_{L^1})]^{\sigma/p'}$$

and the constant C *depends only on* α, p, d.

Proof. Assume first that $||u_0||_{L^\infty} < \infty$ and define the function

$$\tilde{u}(t,x) \;=\; \begin{cases} \Gamma^{-1}(\xi) & , \;\xi \geq 0 \\ 0 & , \;\xi < 0, \end{cases}$$

where $\xi = C_1 t - C_2\,\theta^{1/p'}(x) + C_2\rho$. Then

$$\frac{\partial \tilde{u}}{\partial t} \;-\; \operatorname{div}(\tilde{u}V) \;=\; f,$$

where

$$f \;=\; C_2^{p-1}\,\frac{1}{(p')^{p-1}}\,\tilde{u}(t,x)\,(d-1)\,\theta^{-1/p'} \;\geq\; 0.$$

If

(25) $$\tilde{u}\,|_{t=0} \;\geq\; u|_{t\geq 0}\,,$$

then from the compartial principle as in [8], we obtain

(26) $$\tilde{u} \;\geq\; u.$$

To get (25), we define $C_1 = (C_2/p')^{p-1}C_2$ and C_2 such that $C_2(-R_0+\rho) \geq \Gamma(||u_0||_{L^\infty})$. Then letting $\rho \equiv \rho(\tau) = R_0 + \tau^{1/p}$ and $C_2 = \Gamma(||u_0||_{L^\infty})\,/\,\tau^{1/p}$, we have (26) for $t \in (0,\tau)$ and

$$\operatorname{supp}\, u(t,\cdot) \subset \operatorname{supp}\, \tilde{u}(t,\cdot) \subset \left\{x \,:\, \theta^{1/p'}(x) \;\leq\; R_0 + \tau^{1/p} + \left[\frac{\Gamma(||u_0||_{L^\infty})}{p'\,\tau^{1/p}}\right]^{p-1} t\right\}.$$

So we obtain supp $u(t,\cdot) \subset B^{\theta}_{R(t)}$, where

$$R(t) \;\leq\; R_0 + p^{1/p}\,[\Gamma^{1/p'}(||u_0||_{L^\infty})]\, t^{1/p}. \tag{27}$$

Now if only $||u_0||_{L^1} < \infty$, then $||u(s,\cdot)||_{L^\infty} < \infty$ for $s>0$ since from (22)

$$\Gamma(u(s,x)) \;\leq\; C_1 s^{-\alpha}[\Gamma(||u_0||_{L^1})]^{\sigma}, \quad \sigma \;=\; 1-(p-1)\alpha.$$

Note that (27) has the form

$$R(t) \;\leq\; R_0 + p^{1/p}[\Gamma(||u(s,\cdot)||_{L^\infty})^{p-1}(t-s)]^{1/p}, \quad s<t$$

and iterrating with respect to s, or letting $s = t/2$, we get estimate (24).

Remark 4. In Theorem 3 the dependence of the propagation of support is only on the initial mass of the solution. In the case of equations (3), (4), (5) for $d = 1$ the condition $s\Gamma'(s) \to 0$ for $s \to 0$ is also necessary for the compactness of the support, see for example [1]. The estimate (24) is sharp on the solutions $B_{1,n}$, $n>0$ of the form (6).

For equations (4) and (5) it is possible to get necessary condition on the initial data for the existence of the solution of Cauchy problem in the strip S_T.

Theorem 4 (initial trace). *If $u(t,x)$ is nonnegative, continuous weak solution of (4) or (5) with $a(s) = s^m$, $m>1$ satisfying (2) and $\Gamma(0) = 0$, then there exists a unique nonnegative Borel measure μ such that*

$$\lim_{t\to 0}\int_{R^d} u(t,x)\chi(x)\,dx \;=\; \int_{R^d}\chi(x)\,\mu(dx) \tag{28}$$

for $\chi \in C_0^\infty(\mathbb{R}^d)$ and for $R>0$

$$\frac{1}{|B^{\theta}_{R_0}(x_0)|}\int_{B^{\theta}_{R_0}(x_0)} \mu(dx) \leq C\left\{\left(\frac{T^{1/p}}{R_0}\right)^{d}[u(T,x_0)]^{1/\sigma} + \left(\frac{R_0^p}{T}\right)^{\frac{1}{m(p-1)-1}}\right\}, \tag{29}$$

where C depends only on m, p, α.

Proof. Note that it is sufficient to prove (29) for function u with continuous u_0 with compact support in $B^{\theta}_{R_0}$. Define the function

$$h_\epsilon(x) \;=\; \begin{cases} 1, & \theta(x) \leq R_0 \\ 0, & \theta(x) > R_0+\epsilon \end{cases}$$

for some $\epsilon > 0$, and let $w^\epsilon(t,x)$ be the solution of the problem (4) with $a(s) = s^m$ in S_T and initial condition $w^\epsilon(0,x) = h_\epsilon(x)\,u(o,x)$ in R^d. Then

$$\operatorname{supp}\, w^\epsilon(0,x) \subset B^\theta_{R_0}$$

and $u(t,x) \geq w^\epsilon(t,x)$ in S_T so if we have the estimate (29) for $w^\epsilon(t,x)$, then when $\epsilon \to 0$ we obtain (29) for $u(t,x)$.

If $\operatorname{supp}\, u(0,x) \subset B^\theta_{R_0}$ then from Theorem 3 $\operatorname{supp}\, u(t,x) \subset B^\theta_{R(t)}$ and $R(t)$ has the form (24). Then from (8) for $s = T$, $y = x_0$, integrating the left-hand side, we obtain

$$\Gamma\left(\fint_{B^\theta_{R(t)}(x_0)} u(t,x)\,dx\right) \leq \frac{T^\alpha}{t^\alpha}\left[\Gamma(u(T,x_0)) + C_1\,\frac{R(t)^{\frac{p}{p-1}}}{(T-t)^{\frac{1}{p-1}}}\right],$$

where we denote

$$\fint_{B^\theta_R(x_0)} u(t,x)\,dx = \frac{1}{|B^\theta_R(x_0)|}\int_{B^\theta_R(x_0)} u(t,x)\,dx.$$

Since the conservation mass law takes place, we get with $0<t<T$

$$\tag{30} \begin{aligned} &\Gamma\left(\frac{|B^\theta_{R_0}(x_0)|}{|B^\theta_{R(t)}(x_0)|}\fint_{B^\theta_{R_0}(x_0)} u_0(x)\,dx\right) \\ &\leq \left(\frac{T}{t}\right)^\alpha\left[\Gamma(u(T,x_0)) + C_1\,\frac{R(t)^{\frac{p}{p-1}}}{(T-t)^{\frac{1}{p-1}}}\right]. \end{aligned}$$

Now describe two cases:

a) $\Gamma\left(\fint_{B^\theta_{R_0}(x_0)} u_0(x)\,dx\right) \leq C_1\,\frac{R_0^{\frac{p}{p-1}}}{T^{\frac{1}{p-1}}}$;

b) $\Gamma\left(\fint_{B^\theta_{R_0}(x_0)} u_0(x)\,dx\right) > C_1\,\frac{R_0^{\frac{p}{p-1}}}{T^{\frac{1}{p-1}}}$.

In the case a), the estimate (24) is fulfilled with $C = C_1$. In the case b), we set

$$t = \frac{C_1^{p-1}}{p}\,\frac{R_0^p}{\left[\Gamma\left(\fint_{B^\theta_{R_0}(x_0)} u_0(x)\,dx\right)\right]^{p-1}},$$

then $t < T/p$, $T - t \geq \frac{p-1}{p} T$ and from (24) we obtain $R(t) \leq C_2 R_0$. Then (30) gives

$$C_2^{-dn} \Gamma \left(\fint_{B^{e}_{R_0}(x_0)} u_0(x) dx \right)$$
$$\leq C_3 \left(\frac{T}{R^p}\right)^{\alpha} \left[\Gamma \left(\fint_{B^{e}_{R_0}(x_0)} u_0(x) dx \right) \right]^{1-\sigma} \left[\Gamma(u(T, x_0)) + C_1 C_2^{p'} \left(\frac{R^p}{T}\right)^{\frac{1}{p-1}} \right]$$

i. e. with the correspondence of parameters

$$\Gamma \left(\fint_{B^{e}_{R_0}(x_0)} u_0(x) dx \right) \leq C \left\{ \left(\frac{T}{R_0^p} \right)^{\alpha} \left[\Gamma(u(T, x_0)) + \frac{R_0^{\frac{p}{p-1}}}{T^{\frac{1}{p-1}}} \right]^{\frac{1}{\sigma}} \right\}$$

and (29) holds.

Now let $u(t, x)$ be a nonnegative continuous weak solution in S_T. Then for every $\tau \in (0, T]$

$$\fint_{B^{e}_{R_0}(x_0)} u(\tau, x)\, dx \leq C_4 \Gamma^{-1} \left[\left(\frac{T - \tau}{R_0^p} \right)^{\frac{\alpha}{\sigma}} \Gamma(u(T, x_0))^{\frac{1}{\sigma}} + \frac{R_0^{\frac{p}{p-1}}}{(T - \tau)^{\frac{1}{p-1}}} \right]$$

is fulfilled and C_4 does not depend on τ. Then there exists a sequence $\tau_l \to 0$ such that (28) is fulfilled for every $\chi \in C_0^{\infty}(\mathbb{R}^d)$ and μ satisfies the estimate (29).

The uniqueness is proved in the same way as in [8], [9].

Remark 5. For the equation (3) an analogous result is given in [7], [8] and for (5) with $a(s) = s$ in [10].

4. Regularizing effect

First of all we shall give some sufficient conditions for (2):

1) Let $\psi(q) \in C^3(\mathbb{R}^d)$, $\Gamma(s) \in C(\mathbb{R}) \cap C^2(\mathbb{R} \backslash 0)$, $\Phi(s) = s\Gamma'(s)$. Denote by $\varphi(s)$ the function inverse to $\varphi(s)$ and define

$$(31) \qquad F(s) = \begin{cases} \frac{1}{p-1-(p/d)} |\Phi|^{p-2-(p'/d)} \Phi & \text{for } p' \neq d \\ \operatorname{sign}(\Phi) \ln |\Phi| & \text{for } p' = d. \end{cases}$$

Using the method of [3], [8], we can obtain the next proposition:

Proposition 1. *If there exists a non-convex function $G \in C^2(\mathbb{R})$ such that*

$$GF'' + F' \le (p-2)G'F', \quad G\Phi > 0, \tag{32}$$

then

$$u\varphi'(u)\operatorname{div} V \ge -\frac{G(\varphi(u))}{t}, \quad t>0 \tag{33}$$

for all nonnegative smooth solutions of (1).

Proof. Denote $\psi^{jk} = \frac{\partial^2\psi}{\partial q_j\,\partial q_k}$ and $\psi^{jkl} = \frac{\partial^3\psi}{\partial q_j\,\partial q_k\,\partial q_l}$.
From the homogeneity of ψ it follows that

$$\sum_1^d q_j\,\psi^j = p\psi, \quad \sum_1^d q_k\,\psi^{jk} = (p-1)\psi^j$$

and from the positivity of matrix $D^2\psi = \{\psi^{jk}\}$ we obtain

$$T_r(D^2\psi.Q)^2 \ge \frac{1}{d}\,[T_r(D^2\psi.Q)]^2$$

for every symetric matrix Q.

Denote $\gamma(s) = \Gamma^{-1}(s)$, $g(s) = \gamma(s)/\gamma'(s)$ and let $H(s) > 0$ be a smooth function which we shall choose later. From (1) the function $v = \Gamma(u)$ satisfies the equation

$$v_t = g(v)\sum_1^d \frac{\partial}{\partial x_j}\psi^j(\nabla v) + p\psi(\nabla v) = Hw + p\psi(\nabla v), \tag{34}$$

where we denote

$$w = \frac{g}{H}\sum_1^d \frac{\partial}{\partial x_j}\psi^j(\nabla v).$$

From (34) we obtain

$$w_t = (\frac{g}{H})'\,\frac{H}{g}\,w[Hw + p\psi] + \frac{g}{H}\sum_1^d \frac{\partial}{\partial x_j}\psi^{jk}\,\frac{\partial}{\partial x_k}[Hw + p\psi] \tag{35}$$

and calculating the last two terms on the right side, we get

$$g/H \sum_{j,k=1}^{d} \frac{\partial}{\partial x_j} \psi^{jk} \frac{\partial}{\partial x_k}(Hw) = g \sum_{j,k=1}^{d} \frac{\partial}{\partial x_j} \psi^{jk} \frac{\partial}{\partial x_k} w +$$

$$+ \frac{2gH'}{H} \sum_{j,k=1}^{d} \psi^{jk} v_{x_j} \frac{\partial}{\partial x_k} w + \frac{gH'}{H} w \sum_{j,k=1}^{d} \frac{\partial}{\partial x_j}(\psi^{jk} v_{x_k}) + \frac{gH''}{H} \sum_{j,k=1}^{d} w\psi^{jk} v_{x_j} v_{x_k}$$

$$= g \sum_{j,k=1}^{d} \frac{\partial}{\partial x_j} \psi^{jk} \frac{\partial}{\partial x_k} w + 2g \frac{H'}{H}(p-1) \sum_{k=1}^{d} \psi^k \frac{\partial}{\partial x_k} w +$$

$$+ (p-1)H'w^2 + p(p-1)g \frac{H''}{H} \psi w,$$

$$p\frac{g}{H} \sum_{j,k=1}^{d} \frac{\partial}{\partial x_j} \psi^{jk} \frac{\partial}{\partial x_k} \psi = p\frac{g}{H} \sum_{j,k,l=1}^{d} \frac{\partial}{\partial x_j} \psi^{jk} \psi^l v_{x_k x_l}$$

$$= p \frac{g}{H} \sum_{j,l=1}^{d} \frac{\partial}{\partial x_j} \psi^l \frac{\partial}{\partial x_l} \psi^j$$

$$= p \frac{g}{H} \sum_{j,l=1}^{d} (\frac{\partial}{\partial x_j} \psi^l)(\frac{\partial}{\partial x_l} \psi^j) + p\frac{g}{H} \sum_{j,l=1}^{d} \psi^l \frac{\partial}{\partial x_l} \frac{\partial}{\partial x_j} \psi^j$$

$$= p \frac{g}{H} T_r(D^2\psi.D^2v)^2 + p \frac{g}{H} \sum_{l=1}^{d} \psi^l \frac{\partial}{\partial x_l} (\frac{H}{g} w) \geq$$

$$\geq p \sum_{l=1}^{d} \psi^l \frac{\partial}{\partial x_l} w + p^2 (\frac{H}{g})' \frac{g}{H} \psi w + \frac{p}{d} \frac{g}{H} (\frac{H}{g} w)^2.$$

Now we obtain from (35)

$$w_t \geq \mathcal{L} w, \tag{36}$$

where the differential operator $\mathcal{L}$ has the form

$$\mathcal{L} w = g \sum_{j,k=1}^{d} \frac{\partial}{\partial x_j} \psi^{jk} \frac{\partial}{\partial x_k} w + [2(p-1)H' + p\frac{H}{g}] \sum_{k=1}^{d} \psi^k \frac{\partial}{\partial x_k} w +$$

$$+ p(p-1) \frac{g}{H} (H' + \frac{H}{g})' \psi w + [(p-2)H' + \frac{H}{g} (g' + \frac{p}{d})] w^2.$$

From the other side, we have for $z = -1/t$

$$(37)\qquad z_t - \mathcal{L}z = \frac{1}{t}\,p(p-1)\,\frac{g}{H}\,(H' + \frac{H}{g})'\,\psi + \frac{1}{t^2}\,[1-(p-2)H' - \frac{H}{g}\,(g' + \frac{p}{d})]$$

and if

$$(38)\qquad (H' + \frac{H}{g}) \text{ does not increase and } (p-2)H' + \frac{H}{g}(g' + \frac{p}{d}) \ge 1$$

then we have $\mathcal{L}_t \le \mathcal{L}z$ and $w|_{t=0} > -\infty = z|_{t=0}$.

Using compartion principle for solutions of quasilinear parabolic equations from [5], we obtain (33) with $H(s) = G(\varphi(\gamma(s)))\,/\,\gamma(s)$. Such a choice of H gives (38) since G satisfies (32). To get this we have to set $s = \varphi(\gamma(s))$ and use the properties of φ, γ and Φ.

Remark 6. The assumptions of smoothness of ψ and of the solution u are due to the method of proof. Such conditions are not necessary which is clear from the solutions $B_{1,n}(t,x)$. For such solutions $\Gamma(s)$ is constant, and they are not smooth at the points $\{(t,x) : B_{1,n}(t,x) = 0\}$ and at the points $(t,0)$.

Remark 7. For the equation (5) we give two sufficient conditions on the function $a(s)$ from which there follow the conditions of Proposition 1:

(i) Assume that there exists a constant $k>0$ such that

$$(39)\qquad k\,\{a\,[sa'' + (\frac{p}{d} + 2 - p)\,a'] + (p-2)s\,a'^{\,2}\} \ge s\,a'^{\,2}.$$

Then $G(s) = ks$ satisfies (32). For $p = 2$ the condition (39) is equal to the conditions in [3], [4], [8], where (33) is obtained.

(ii) Assume that $a(s)$ satisfies

$$(40)\quad (1 - \frac{p'}{d})\,[sa'' + \frac{p'}{d}\,a']\int_0^s \lambda^{(p'/d)-1}\,a'(\lambda)\,d\lambda \ge (1 - \frac{p'}{d})\,s^{p'/d}\,a'^{\,2}(s).$$

Then we may choose

$$G(s) = \frac{1}{p-1}\,F^{\frac{1}{p-1}}(s)\int_{s_0}^{s} F^{-\frac{1}{p-1}}(\xi)\,d\xi,$$

where $F(s)$ is defined by (32). Now for $p'/d < 1$, (33) takes the form

$$\sum_1^d \frac{\partial}{\partial x_j}\,|\frac{\partial a(u)}{\partial x_j}|^{p-2}\,\frac{\partial a(u)}{\partial x_j} \ge -\frac{1}{t\,u^{p'/d}\,a'(u)}\int_{s_0}^{u}\frac{\lambda^{p'/d}\,a'(\lambda)}{\lambda}\,d\lambda.$$

For the equation (4) with $a(s) = s^m$, the functions Γ and φ take the form

$$\Gamma(s) = \frac{m}{m+1-p'} |s|^{m+1-p'}, \qquad \varphi(s) = \frac{m(p-1)}{(m+1)(p-1)-1} \, s \, |s|^{\frac{m(p-1)-1}{p-1}}.$$

From (i) and (ii) for (32) we obtain

$$\sum_1^d \frac{\partial}{\partial x_j} |\frac{\partial u^m}{\partial x_j}|^{p-2} \frac{\partial u^m}{\partial x_j} \geq - \frac{d}{[m(p-1)-1]d+p} \frac{1}{t}$$

which becomes the equality in the sense of $\mathcal{D}'(R^d)$ on the solutions $B_{1,n}(t,x)$, $n = m+1-p'$, see Remark 6.

2) The next sufficient condition for regularizing effect use some apriori information on the velosity V with components $\psi^j(\nabla\Gamma(u))$ in the equation (1) — to be Lipschitz continuous on spatial variables. In the previous works the smoothness was obtained dealing naturally with the solution but not with combinations of solution's derivatives, like velosity. It is clear that the last is the smoothness object in the domain of positivity of the solution.

Proposition 2. *Let u be a continuous weak solution of (1) such that the velosity $V = \{\psi^j(\nabla u)\}$, $v = \Gamma(u)$ satisfies*

$$|V(t,x) - V(t,y)| \leq \frac{1}{\nu t} |x-y| \tag{41}$$

in S_T, where ν is a constant $\nu > 0$ and depends only on p, d. Then

$$div V \geq -\frac{1}{\nu t} \tag{42}$$

in the sense of $\mathcal{D}'(\mathbb{R}^d)$.

Proof. Fix x and define the function

$$F(t,x,y) = v(t,y) + \nu_0 t \, \theta(\frac{y-x}{\nu_0 t} - V(t,x)), \quad 0 < \nu_0 < \nu,$$

then $F(t,x,y) \geq 0$. Since $\theta(z)$ is conjugate to $\psi(q)$, we get $\theta(\psi^j(q)) = (p-1)\psi(q)$ and $\psi^j(\nabla\theta(z)) = z_j$. Then in the point of local minimum

$$0 = F_{y_j} = v_{y_j} + \theta^j(\frac{y-x}{\nu_0 t} - V(t,x)) = \theta^j(V(t,y)) + \theta^j(\frac{y-x}{\nu_0 t} - V(t,x)),$$

which gives

$$V(t,y) + \frac{y}{\nu_0 t} = V(t,x) + \frac{x}{\nu_0 t}. \tag{43}$$

Since V satisfies (42) then (43) has a unique solution $y = x$. In such points the condition that $F(t, x, y)$ has a local minimum gives

$$D^2 v + \frac{1}{\nu_0 t} D^2 \theta(-V) \geq 0. \tag{44}$$

Multiplying (44) by $D^2\psi(\nabla v)$ which is a nonnegative defined matrix, we obtain

$$T_r(D^2\psi . D^2 v + D^2\psi \cdot \frac{1}{\nu_0 t} D^2\theta) \geq 0,$$

which gives (42).

3) For some classes of equations (1) there are results on existence of solution obtained with regularizing procedure, a compartial principle and a regularizing effect. Such results are obtained for (5), $d = 1$ with $a(s) = s$ in [14] and $a(s) = s^m$ in [12]. In the case $d > 1$ for the equation (5) the regularizing effect was obtained in [13] for $a(s) = s$ and in [15] for $a(s) = s^m$.

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