Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New series Vol.8, 1994, Fasc.1

Results on the Non-Commutative Neutrix Product of Distributions

Brian Fisher †, *Ekrem Savaş* ‡,
*Serpil Pehlivan**, *Emin Özçağ* †

Presented by Bl. Sendov

The neutrix product of the distributions x_-^{-r} and x_+^s is evaluated for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$. Further neutrix products are then deduced.

In the following, we let N be the neutrix, see J. G. van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n: \qquad \lambda > 0, \quad r = 1, 2, \ldots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

We now let $\rho(x)$ be any infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [2] or [4].

Definition 1. Let f and g be distributions in $\mathcal{D}'$ for which on the interval (a, b), f is the k-th derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$ is a locally summable function in $L^q(a, b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [F g^{(i)}]^{(k-i)}.$$

The following definition for the neutrix product of two distributions was given in [5] and generalizes Definition 1.

Definition 2. Let f and g be distributions in $\mathcal{D}'$ and let $g_n(x) = (g * \delta_n)(x)$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to the distribution h on the interval (a, b) if

$$\mathop{\mathrm{N-lim}}_{n \to \infty} \langle f(x) g_n(x), \phi(x) \rangle = \langle h(x), \phi(x) \rangle$$

for all functions ϕ in $\mathcal{D}$ with support contained in the interval (a, b).

Note that if

$$\lim_{n \to \infty} \langle f(x) g_n(x), \phi(x) \rangle = \langle h(x), \phi(x) \rangle,$$

we simply say that the *product* $f.g$ exists and equals h, see [4].

It is obvious that if the product $f.g$ exists then the neutrix product $f \circ g$ exists and $f.g = f \circ g$. Further, it was proved in [4] that if the product fg exists by Definition 1 then the product $f.g$ exists by Definition 2 and $fg = f.g$. Note also that although the product defined in Definition 1 is always commutative, the product and neutrix product defined in Definition 2 is in general non-commutative.

The following theorem holds, see [5].

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix products $f \circ g$ and $f \circ g'$ (or $f' \circ g$) exist on the interval (a,b). Then the neutrix product $f' \circ g$ (or $f \circ g'$) exists on the interval (a,b) and*

$$(f \circ g)' = f' \circ g + f \circ g'$$

on the interval (a,b).

We now prove the following extension of Theorem 1.

Theorem 2. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix products $f \circ g^{(i)}$ (or $f^{(i)} \circ g$) exist on the interval (a,b) for $i = 0,1,2,\ldots,r$. Then the neutrix products $f^{(k)} \circ g$ (or $f \circ g^{(k)}$) exist on the interval (a,b) for $k = 1,2,\ldots,r$ and*

$$f^{(k)} \circ g = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [f \circ g^{(i)}]^{(k-i)} \tag{1}$$

or

$$f \circ g^{(k)} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [f^{(i)} \circ g]^{(k-i)} \tag{2}$$

on the interval (a,b) for $k = 1,2,\ldots,r$.

Proof. The theorem is true by Theorem 1 for the case $r = 1$ and so suppose the theorem is true for some r and that the neutrix products $f \circ g^{(i)}$ exist for $i = 0,1,2,\ldots,r+1$. Then by the assumption, the neutrix product $f^{(k)} \circ g$ exists and then by Theorem 1, the neutrix product $f^{(k+1)} \circ g$ exists and

$$\begin{aligned}
[f^{(k)} \circ g]' &= f^{(k+1)} \circ g + f^{(k)} \circ g' \\
&= f^{(k+1)} \circ g + \sum_{i=0}^{k} \binom{k}{i} (-1)^i [f \circ g^{(i+1)}]^{(k-i)} \\
&= \sum_{i=0}^{k} \binom{k}{i} (-1)^i [f \circ g^{(i)}]^{(k-i+1)}
\end{aligned}$$

and so

$$\begin{aligned}
f^{(k+1)} \circ g &= \sum_{i=0}^{k} \binom{k}{i} (-1)^i [f \circ g^{(i)}]^{(k-i+1)} \\
&\quad + \sum_{i=1}^{k+1} \binom{k}{i-1} (-1)^i [f \circ g^{(i)}]^{(k-i+1)} \\
&= \sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^i [f \circ g^{(i)}]^{(k-i+1)}.
\end{aligned}$$

The result of equation (1) now follows by induction.
The proof of equation (2) follows similarly.

The next theorem was proved in [6].

Theorem 3. *The neutrix products* $\ln x_- \circ \delta^{(r)}(x)$ *and* $\delta^{(r)}(x) \circ \ln x_-$ *exist and*

$$\ln x_- \circ \delta^{(r)}(x) = [c(\rho) + \tfrac{1}{2}\psi(r)]\delta^{(r)}(x), \tag{3}$$

$$\delta^{(r)}(x) \circ \ln x_- = c(\rho)\delta^{(r)}(x) \tag{4}$$

for $r = 0, 1, 2, \ldots$, *where*

$$c(\rho) = \int_0^1 \ln t\rho(t)\,dt$$

and

$$\psi(r) = \begin{cases} 0, & r = 0, \\ \displaystyle\sum_{i=1}^{r} i^{-1}, & r \geq 1. \end{cases}$$

It was shown in [6] that by suitable choice of the function ρ, $c(\rho)$ can take any negative value.

We now define the distributions x_+^{-r}, x_-^{-r}, $F(x_+, -r)$ and $F(x_-, -r)$ for $r = 1, 2, \ldots$ by

$$(r-1)!x_+^{-r} = (-1)^{r-1}(\ln x_+)^{(r)}, \quad (r-1)!x_-^{-r} = -(\ln x_-)^{(r)},$$

$$\langle F(x_+, -r), \phi(x)\rangle = \int_0^\infty x^{-r}\Big[\phi(x) - \sum_{i=0}^{r-2}\frac{x^i}{i!}\phi^{(i)}(0) - \frac{x^{r-1}}{(r-1)!}\phi^{(r-1)}(0)H(1-x)\Big]\,dx,$$

$$\langle F(x_-, -r), \phi(x)\rangle = \int_0^\infty x^{-r}\Big[\phi(-x) - \sum_{i=0}^{r-2}\frac{(-x)^i}{i!}\phi^{(i)}(0) - \frac{(-x)^{r-1}}{(r-1)!}\phi^{(r-1)}(0)H(1-x)\Big]\,dx$$

for arbitrary ϕ in $\mathcal{D}$, where H denotes Heaviside's function.

Note that the distributions $F(x_+, -r)$ and $F(x_-, -r)$ we have just defined were used by I. M. Gel'fand and G. E. Shilov [8] to denote the distributions x_+^{-r} and x_-^{-r} respectively.

It was proved in [3] that

$$x_+^{-r} = F(x_+, -r) + \frac{(-1)^r\psi(r-1)}{(r-1)!}\,\delta^{(r-1)}(x), \tag{5}$$

$$x_-^{-r} = F(x_-, -r) - \frac{\psi(r-1)}{(r-1)!}\,\delta^{(r-1)}(x) \tag{6}$$

for $r = 1, 2, \ldots$.

It then follows that

$$x^{-r} = x_+^{-r} + (-1)^r x_-^{-r} = F(x_+, -r) + (-1)^r F(x_-, -r) \tag{7}$$

for $r = 1, 2, \ldots$.

Some of the results obtained in the following theorems were first obtained in [7], but by making use of Theorem 2 the proofs are simplified considerably.

Theorem 4. *The neutrix products $x_-^{-r} \circ x_+^s$ and $x_+^s \circ x_-^{-r}$ exist and*

$$x_-^{-r} \circ x_+^s = x_-^{-r} x_+^s = 0, \tag{8}$$

$$x_+^s \circ x_-^{-r} = x_+^s x_-^{-r} = 0 \tag{9}$$

for $r = 1, 2, \ldots$ and $s = r, r+1, \ldots$ and

$$x_-^{-r} \circ x_+^s = \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^{i-1} s!}{(r-1)!} [c(\rho) + \tfrac{1}{2}\psi(i-s-1)] \delta^{(r-s-1)}(x), \tag{10}$$

$$x_+^s \circ x_-^{-r} = \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^{i-1} s!}{(r-1)!} c(\rho) \delta^{(r-s-1)}(x) \tag{11}$$

for $r = 1, 2, \ldots$ and $s = 0, 1, \ldots, r-1$.

Proof. The product of the functions $\ln x_-$ and x_+^s is just a straightforward product of functions in $L^2(a, b)$ for every bounded interval (a, b) and so

$$\ln x_- \circ x_+^s = \ln x_- x_+^s = 0 \tag{12}$$

for $s = 0, 1, 2, \ldots$. Putting $g(x) = x_+^s$, we have

$$g^{(i)}(x) = \begin{cases} \dfrac{s!}{(s-i)!} x_+^{s-i}, & 0 \le i \le s, \\ s!\delta^{(i-s-1)}(x), & i > s. \end{cases}$$

Thus, by equation (12) we have

$$\ln x_- \, g^{(i)}(x) = 0$$

for $i = 0, 1, \ldots, s$ and by equation (3) we have

$$\ln x_- \circ g^{(i)}(x) = s![c(\rho) + \tfrac{1}{2}\psi(i-s-1)]\delta^{(i-s-1)}(x)$$

for $i = s+1, s+2, \ldots$. It now follows from equation (1) that

$$
\begin{aligned}
(\ln x_-)^{(r)} \circ g(x) &= -(r-1)! x_-^{-r} \circ x_+^s \\
&= \sum_{i=0}^{r} \binom{r}{i} (-1)^i [\ln x_- \circ g^{(i)}(x)]^{(r-i)} \\
&= \begin{cases} 0, & r \le s, \\ \displaystyle\sum_{i=s+1}^{r} \binom{r}{i} (-1)^i s! [c(\rho) + \tfrac{1}{2}\psi(i-s-1)] \delta^{(r-s-1)}(x), & r > s. \end{cases}
\end{aligned}
$$

Equations (8) and (10) now follow immediately.

Equations (9) and (11) follow similarly using equation (2) and (4).

Corollary 1. *The neutrix products $x_+^{-r} \circ x_-^s$ and $x_-^s \circ x_+^{-r}$ exist and*

$$x_+^{-r} \circ x_-^s = x_+^{-r} x_-^s = 0, \tag{13}$$

$$x_-^s \circ x_+^{-r} = x_-^s x_+^{-r} = 0 \tag{14}$$

for $r = 1, 2, \ldots$ and $s = r, r+1, \ldots$ and

$$x_+^{-r} \circ x_-^s = \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^{r+s+i} s!}{(r-1)!} [c(\rho) + \tfrac{1}{2}\psi(i-s-1)] \delta^{(r-s-1)}(x), \tag{15}$$

$$x_-^s \circ x_+^{-r} = \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^{r+s+i} s!}{(r-1)!} c(\rho) \delta^{(r-s-1)}(x) \tag{16}$$

for $r = 1, 2, \ldots$ and $s = 0, 1, \ldots, r-1$.

Proof. Equations (13), (14), (15) and (16) follow on replacing x by $-x$ in equations (8), (9), (10) and (11) respectively.

Theorem 5. *The neutrix products $x_+^{-r} \circ x_+^s$ and $x_+^s \circ x_+^{-r}$ exist and*

$$x_+^{-r} \circ x_+^s = x_+^{-r} x_+^s = x_+^{s-r}, \tag{17}$$

$$x_+^s \circ x_+^{-r} = x_+^s x_+^{-r} = x_+^{s-r} \tag{18}$$

for $r = 1, 2, \ldots$ and $s = r, r+1, \ldots$ and

$$
\begin{aligned}
x_+^{-r} \circ x_+^s = x_+^{-r+s} &- \frac{(-1)^{r+s}}{(r-s-1)!} [\psi(r-1) + \psi(r-s-1)] \delta^{(r-s-1)}(x) \\
&- \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^{r+i} s!}{(r-1)!} [c(\rho) + \tfrac{1}{2}\psi(i-s-1)] \delta^{(r-s-1)}(x),
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
x_+^s \circ x_+^{-r} = x_+^{-r+s} &- \frac{(-1)^{r+s}}{(r-s-1)!} [\psi(r-1) + \psi(r-s-1)] \delta^{(r-s-1)}(x) \\
&- \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^{r+i} s!}{(r-1)!} c(\rho) \delta^{(r-s-1)}(x)
\end{aligned} \tag{20}
$$

for $r = 1, 2, \ldots$ *and* $s = 0, 1, \ldots, r-1$.

Proof. It is easily proved that the product of the distribution $F(x_+, -r)$ and the infinitely differentiable function x^s is given by

$$F(x_+, -r)\, x^s = x^s F(x_+, -r) = x_+^{s-r} \tag{21}$$

for $r = 1, 2, \ldots$ and $s = r, r+1, \ldots$ and

$$F(x_+, -r)\, x^s = x^s F(x_+, -r) = F(x_+, -r+s) \tag{22}$$

for $r = 1, 2, \ldots$ and $s = 0, 1, \ldots, r-1$.

Since the neutrix product is clearly distributive with respect to addition, it follows that

$$\begin{aligned} x_+^{-r}\, x^s &= x_+^{-r}\,[x_+^s + (-1)^s x_-^s] \\ &= x_+^{-r} \circ x_+^s + (-1)^s x_+^{-r} \circ x_-^s \\ &= \left[F(x_+, -r) - \frac{(-1)^r \psi(r-1)}{(r-1)!}\,\delta^{(r-1)}(x)\right] x^s \\ &= \begin{cases} x_+^{s-r}, & s \geq r, \\ F(x_+, -r+s) - \dfrac{(-1)^{r+s}\psi(r-1)}{(r-s-1)!}\delta^{(r-s-1)}(x), & s < r. \end{cases} \end{aligned}$$

Equations (17) and (19) now follow immediately on using equations (5), (13) and (15), the product $x_+^{-r}\, x^s$ existing when $s \geq r$.

Equations (18) and (20) follow similarly on using equations (6), (14) and (16).

Corollary 1. *The neutrix products* $x_-^{-r} \circ x_-^s$ *and* $x_-^s \circ x_-^{-r}$ *exist and*

$$x_-^{-r} \circ x_-^s = x_-^{-r}\, x_-^s = x_-^{s-r}, \tag{23}$$

$$x_-^s \circ x_-^{-r} = x_-^s\, x_-^{-r} = x_-^{s-r} \tag{24}$$

for $r = 1, 2, \ldots$ *and* $s = r, r+1, \ldots$ *and*

$$\begin{aligned} x_-^{-r} \circ x_-^s &= x_-^{-r+s} + \frac{1}{(r-s-1)!}[\psi(r-1) + \psi(r-s-1)]\delta^{(r-s-1)}(x) \\ &+ \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^{s+i} s!}{(r-1)!}[c(\rho) + \tfrac{1}{2}\,\psi(i-s-1)]\delta^{(r-s-1)}(x), \end{aligned} \tag{25}$$

$$\begin{aligned} x_-^s \circ x_-^{-r} &= x_-^{-r+s} + \frac{1}{(r-s-1)!}[\psi(r-1) + \psi(r-s-1)]\delta^{(r-s-1)}(x) \\ &+ \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^{s+i} s!}{(r-1)!} c(\rho)\delta^{(r-s-1)}(x) \end{aligned} \tag{26}$$

for $r = 1, 2, \ldots$ *and* $s = 0, 1, \ldots, r-1$.

Proof. Equations (23), (24), (25) and (26) follow on replacing x by $-x$ in equations (17), (18), (19) and (20) respectively.

Corollary 2. *The neutrix products* $x^{-r} \circ x_+^s$, $x^{-r} \circ x_-^s$, $x_+^s \circ x^{-r}$ *and* $x_-^s \circ x^{-r}$ *exist and*

$$x^{-r} \circ x_+^s = x^{-r} xs = x_+^{s-r}, \tag{27}$$

$$x^{-r} \circ x_-^s = x^{-r} x_-^s = (-1)^r x_-^{s-r}, \tag{28}$$

$$x_+^s \circ x^{-r} = x_+^s x^{-r} = x_+^{s-r}, \tag{29}$$

$$x_-^s \circ x^{-r} = x_-^s x^{-r} = (-1)^r x_-^{s-r} \tag{30}$$

for $r = 1, 2, \ldots$ *and* $s = r, r+1, \ldots$ *and*

$$\begin{aligned} x^{-r} \circ x_+^s = x_+^{-r+s} &- \frac{(-1)^{r+s}}{(r-s-1)!}[\psi(r-1) + \psi(r-s-1)]\delta^{r-s-1)}(x) \\ &- \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^{r+i} s!}{(r-1)!}[2c(\rho) + \psi(i-s-1)]\delta^{(r-s-1)}(x), \end{aligned} \tag{31}$$

$$\begin{aligned} x^{-r} \circ x_-^s = (-1)^r x_-^{-r+s} &+ \frac{(-1)^r}{(r-s-1)!}[\psi(r-1) + \psi(r-s-1)]\delta^{(r-s-1)}(x) \\ &+ \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^{r+s+i} s!}{(r-1)!}[2c(\rho) + \psi(i-s-1)]\delta^{(r-s-1)}(x), \end{aligned} \tag{32}$$

$$\begin{aligned} x_+^s \circ x^{-r} = x_+^{-r+s} &- \frac{(-1)^{r+s}}{(r-s-1)!}[\psi(r-1) + \psi(r-s-1)]\delta^{(r-s-1)} \\ &- \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^{r+i} s!}{(r-1)!} 2c(\rho)\delta^{(r-s-1)}(x), \end{aligned} \tag{33}$$

$$\begin{aligned} x_-^s \circ x^{-r} = (-1)^r x_-^{-r+s} &+ \frac{(-1)^r}{(r-s-1)!}[\psi(r-1) + \psi(r-s-1)]\delta^{(r-s-1)}(x) \\ &+ \sum_{i=s+1}^{r} \binom{r}{i} \frac{(-1)^{r+s+i} s!}{(r-1)!} 2c(\rho)\delta^{(r-s-1)}(x) \end{aligned} \tag{34}$$

for $r = 1, 2, \ldots$ *and* $s = 0, 1, \ldots, r-1$.

Proof. Equations (27) and (29) follow from equations (7), (8), (9), (17) and (18). Equations (28) and (30) follow on replacing x by $-x$ in equations (27) and (29) respectively. Equations (31) and (33) follow from equations (7), (10), (11), (19) and (20). Equations (32) and (34) follow from equations (31) and (33) on replacing x by $-x$ in equations (31) and (33) respectively.

1. References

[1] J.G. van der Corput. Introduction to the neutrix calculus. *J. Analyse Math.*, 7(1959-60), 291-398.
[2] B. Fisher. The product of distributions. *Quart. J. Math. Oxford* (2), **22**, 1971, 291-298.
[3] B. Fisher. Some notes on distributions. *Math. Student*, **48**, 1980, 269-281.
[4] B. Fisher. On defining the product of distributions. *Math. Nachr.*, **99**, 1980, 239-249.
[5] B. Fisher. A non-commutative neutrix product of distributions. *Math. Nachr.*, **108**, 1982, 117-127.
[6] B. Fisher. Some results on the non-commutative neutrix product of distributions, *Trabajos de Matematica*, **44**, Buenos Aires, 1983.
[7] B. Fisher, M. Itano. A note on the non-commutative neutrix product of distributions. *Mem. Fac. Int. Arts Sci.*, Hiroshima Univ., **13**, 1987, 1-11.
[8] I.M. Gel'fand, G.E. Shilov. Generalized Functions. Vol. I, Academic Press, 1964.

† Department of Mathematics,
Leicester University,
Leicester, LE1 7RH, ENGLAND.

‡ Department of Mathematics,
Firat University,
Elaziğ, TURKEY.

* Akdeniz University,
Isparta Engineering Faculty,
32001 Isparta, TURKEY.

Received 27.09.1992