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Modeling of Images

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Using the notion completed graph of a bounded function, which is a closed and bounded point set in the three dimentional Euclidean space R_3, and exploring the **Hausdorff distance** between these sets, a metric space IM_D of functions is defined. It is shown that the functions $f \in IM_D$, defined on the square $D = [0,1]^2$, are appropriate mathematical models of real world images.
The metric space IM_D contains pixel functions which are produced through digitizing images. It is proved that every function $f \in IM_D$ may be digitized and represented by a pixel function p_n, with n pixels, in such a way, that the distance between f and p_n is no greater than $2n^{-1/2}$.
We claim that the Hausdorff distance is the most natural distance to measure the difference between two pixel representation of a given image. This gives a natural mathematical measure for the quality of the compression produced through different methods. An $O(n \log n)$ algorithm is proposed for calculating the image of the Hausdorff difference between two images represented with n pixels each.

2. Introduction

Every image is represented through a so called pixel function p_n, with n pixels, i. e., p_n is fully defined by n given numbers.

The image compression is an approximation procedure in which a pixel function with n pixels is approximated by a function f, such that each value of f may be calculated from m given numbers. Using f, one produces a pixel function p_n^*, offered as a replacement of p_n. The goal of the image compression is to maximize the ratio $\frac{n}{m}$ in keeping the "quality" of reproduction of p_n through p_n^*. This "quality" is measured usually in an expert way - "how it appears".

The problem of building a mathematical model of real world images, was studied extensively by M. F. Barnsley [1], F. M. Barnsley and L. P. Hurd [2] and others. In addition to the eight basic properties of the set $\Re$ of all real world images, given in [2] we add four more. Let a set IM_D of functions of two variables, defined on the unit square $D = [0,1]^2$, be used as a mathematical model of images[2], then it will be desirable that the set IM_D has also the following properties.

i)The set $IM_{D(s)}$ of pixel functions, with $n = 2^{2s}$ pixels, has to be a subset of IM_D.

ii) The set IM_D has to be metrized by a metric r in such a way that for every element $f \in IN_D$ and every natural s a pixel function $p_n(f)$ exists such that

$$(2.1) \qquad r(f, p_n(f)) \leq \varrho(s), \quad \text{where} \quad \lim_{s \to \infty} \varrho(s) = 0$$

and the function ϱ does not depend on the function f.

iii) The metric space (IM_D, r) has to be complete.

iv) If two pixel functions are very close with respect to the distance r, then the images represented by these two functions have to be "very similar".

The condition, that the function ϱ in (2.1) does not depend on the function f, may be called "principle of uniform digitizing". The pixel functions used in the practice of digitizing are within very small range of s, (usually $7 \leq s \leq 11$). That means that the principle of uniform digitizing is valid. If the principle of uniform digitizing is not valid, then there will exist a number $\epsilon > 0$, such that for every natural s_0, there will exist a real world image, modelled by a function f with $r(f, p_n) \geq \epsilon$, for every pixel function with $n = 2^{2s_0}$ pixels.

Let us mention, that the classical L_p spaces have the property ii) for very restricted sets of functions, which are not rich enough to model real world images.

A successful mathematical model for real world images are the fractals [2]. The most natural distance in the fractal geometry is the Hausdorff distance. The Hausdorff distance has been used also in approximation of functions [8], [9], [10]. To obtain completeness in a functional space, wider than the space of continuous functions, metrized by the Hausdorff distance, one needs to "complete" the graphs of the functions or to consider segment valued functions.

The function spaces topologized by Hausdorff distance are not Banach spaces. This makes them peculiar and more difficult to use. Our purpose in this paper is to overcome the prejudices and to show the usefulness of functional spaces, metrized by Hausdorff distance, for mathematical modeling of real world images.

In section 2 we consider the set B_Ω of all bounded single valued functions, defined in a compact Ω and the set SC_Ω of all, so called *segment continuous* functions, defined in Ω, which are segment valued and their graphs are closed and bounded. We say that $f \subset g$ for $f \in B_\Omega$, $g \in SC_\Omega$ if $f(x) \in g(x)$ for all

[2] That means to use the model $\Re_{ii}$ of F. M. Barnsley and L. P. Hurd [2].

$x \in \Omega$ and define the completed graph $F(f;x) = [I(f;x), S(f;x)]$ of a function f from B_Ω as the intersection of the graphs of all functions from SC_Ω containing the function f. We factorize the set B_Ω, considering two functions not different (equivalent) if they have equale completed graphs.

We call a function f Hausdorff continuous if every other function with the same completed graph is equivalent to f. The set of the H-continuous functions, defined on the compact Ω, is denoted by HC_Ω. The complited graphs of the functions from HC_Ω are a subset of SC_Ω. The main advantage of the set HC_Ω is that it is a linear space.

We metrize the defined functional spaces, using a metric $\rho(x,\xi)$ between two points x,ξ of the compact Ω. First we introduce a metric in the space $\mathbf{R}_\Omega = \mathbf{R}\times\Omega$, where $\mathbf{R}$ is the set of the real numbers, as follows

$$\rho_\alpha((x,z),(\xi,\zeta)) = \max\{|z-\zeta|, \alpha^{-1}\rho(x,\xi)\},$$

where $x,\xi \in \mathbf{R}$, $z,\zeta \in \Omega$.

The completed graph of every bounded function, defined on Ω, is a closed and bounded subset of $\mathbf{R}_\Omega$. We define the Hausdorff distance $r(\alpha, f, g)$ with parameter α between two functions as the Hausdorff distance between their completed graphs, considered as subsets of points in the space $\mathbf{R}_\Omega$. The functional spaces $B_\Omega, SC_\Omega, HC_\Omega$, metrized with the Hausdorff metric $r(\alpha; f, g)$ are denoted respectively with $B^\alpha_\Omega, SC^\alpha_\Omega, HC^\alpha_\Omega$.

In section 3 it is proved that SC^α_Ω is a complete space in the case $\Omega = D = [0,1]^2$. The metric space HC^α_Ω is not complete, as in the case of continuous functions with uniform norm. Using the Hausdoff metric, we define the modulus of H-continuity $\tau(\alpha, f;\delta)$ such that the necessary and sufficient condition for the function f to be H-continuous is

$$\lim_{\delta\to+0} \tau(\alpha, f;\delta) = \tau(\alpha, f;0) = 0.$$

As we have to expect, the subsets of H-equicontinuous functions in HC_D are complete.

We denote by $IM^\alpha_D \subset HC^\alpha_D$ the set of single valued and upper semicontinuous functions ($f(x) = S(f;x)$). A function $f \in IM^\alpha_D$ is called an image function of class ν if

$$\tau(\alpha, f;\delta \le \nu\delta/\alpha.$$

The space of image functions of class ν is denoted by $IM^{\alpha,\nu}_D$ and it is proved that this space is complete.

In section 4 we study the properties of the space $IM^{\alpha,\nu}_D$ and give evidents in support of our claim that this space is a very natural mathematical model of the real world images.

In this section we consider all types of contractive operators used in fractal image compression and prove, that if such an operator has the contractivity factor λ it is an operator in $IM_D^{\alpha,\nu}$ if $\lambda \leq 1 - \frac{1}{\nu}$. This solves the theoretical problem of the convergence with the fractal transforms for grayscale image compression.

3. Sets of functions

3.1 Notations and definitions

Let Ω be a compact, $\mathbf{R}$ be the set of points on the real line (the set of the reals) and $[\mathbf{R}]$ be the set of all segmets $[a, b]$, $a, b \in \mathbf{R}$ of real numbers. We shall consider single valued functions $f : \Omega \to \mathbf{R}$ and segment valued functions $f : \Omega \to [\mathbf{R}]$. The function f is called bounded, if its uniform norm

$$\|f\|_\Omega = \|f\| = \sup_{x\in\Omega}\{|z| : \ z \in f(x)\} \leq \infty,$$

is finite. We shall be interested on applications in image processing and therefore will pay special attention to the case when Ω is the unit square $D = \{(x, y) : x, y \in [0,1]\}$.

We will use the notations:

B_Ω - the set of all bounded real functions f, defined on Ω,

C_Ω - the set of all continuous real functions f, defined on Ω,

BS_Ω - the set of all bounded segment functions f, defined on Ω.

The value $f(x)$ of a segment function f is a segment $[\underline{f}(x), \overline{f}(x)]$, where $\underline{f}, \overline{f} \in B_\Omega$.

SC_Ω - the subset of all $f \in BS_\Omega$ with compact graphs. For $\Omega = D$ the set SC_D is the set of all bounded and closed subsets of the three dimentional Euclidean space R_3, such that if $F \in SC_D$ then the projection of F on the plane (x, y) coincides with D and F is convex toward the axis z orthogonal to the plane (x, y). The functions from SC_Ω, for reason to be explained later, are called *segment continuous*.

Let ρ be a distance in Ω. In the case $\Omega = D$ we shall use the square distance

$$\rho((x, y), (\xi, \eta)) = \max[|x - \xi|, |y - \eta|]. \tag{3.2}$$

We define the *extended Baire functions* for every segment function $f \in BS_\Omega$ and hence for every function $f \in B_\Omega$

$$I(\Omega, \delta, f; x) = I(\delta, f; x) = \inf\{z : \ z \in f(u); \ u \in O(x, \delta) \cap \Omega\},$$

$$S(\Omega, \delta, f; x) = S(\delta, f; x) = \sup\{z : \ z \in f(u); \ u \in O(x, \delta) \cap \Omega\},$$

where $O(u,\delta) = \{x : \rho(x,u) < \delta\}$, and the *Baire functions*

$$I(\Omega, f; x) = I(f; x) = \lim_{\delta\to\infty} I(\delta, f; x),$$

$$S(\Omega, f; x) = S(f; x) = \lim_{\delta\to\infty} S(\delta, f; x, y).$$

It is directly seen that for the *modulus of continuity*

$$\omega(f;\delta) = \sup_{x\in D}\{S(\delta/2, f; x) - I(\delta/2, f; x)\}$$

is valid.

Let G be a subset of $\Omega\times\mathbf{R}$. We say that $f \subset G$ if $(x, f(x)) \in G$ for every $x \in \Omega$.

Definition 2.1. The completed graph [10] of a function $f \in B_\Omega$ is the segment function $F(f) \in BS_\Omega$, defined by

$$F(f;x) = [I(f;x), S(f;x)].$$

It is easy to see that

$$F(f) = \bigcap_{G\in SC_D, f\subset G} G,$$

and that for every $f \in B_\Omega$, $F(f) \in SC_\Omega$, $\|F(f)\| = \|f\|$.

Two functions $f, g \in B_\Omega$ shall be considered not different, $f \approx g$, if $F(f) = F(g)$. This divides B_Ω in classes of equivalence.

If $f \in BS_\Omega$, then $I(f) \in B_\Omega$ (resp. $S(f) \in B_\Omega$) is lower (resp. upper) semicontinuous. Let us recall that a function $f \in B_\Omega$ is lower (resp. upper) semicontinuous if and only if there exists a sequence $\{f_n\}_1^\infty$ in C_Ω such that for each n $f_n(x) \le f(x)$ (resp. $f_n(x) \ge f(x)$) and $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \Omega$.

Definition 2.2. To every segment valued function $f \in SB_\Omega$ there corresponds a single valued function $\hat{f} \in B_\Omega$ defined in the following way

$$\hat{f}(x) = \frac{1}{2}(I(f;x) + S(f;x)) \text{ for every } x \in \Omega.$$

A single valued function $f \in B_\Omega$ is called normal if

$$f(x) = \hat{f}(x) \text{ for every } x \in \Omega.$$

From the definition of the normal functions and the Lebesgue integrability of the semicontinuos functions, we have

3.2 Segment continuous functions

Let us consider certain properties of the functions from SC_Ω, in the case $\Omega = D$, which justify the name segment continuous. We first prove an intermediate value property.

Lemma 2.1. *If $f \in SC_D$ and for $(x_1, y_1), (x_2, y_2) \in D$ and $z_1 = f(x_1, y_1)$ and $z_2 = f(x_2, y_2)$, and the inequality $z_1 < z_2$ holds, then for each $z_0 \in (z_1, z_2)$ there exists $t_0 \in [0, 1]$ such that $z_0 \in f(x_1 + (x_2 - x_1)t_0, y_1 + (y_2 - y_1)t_0)$.*

Proof. If $S(f; x_1, y_1) \geq z_0$, then we can set $t_0 = 0$. Otherwise,

$$S(f; x_1, y_1) < z_0 \leq S(f; x_2, y_2)$$

and we set

$$t_0 = \sup\{t : t \in [0,1], S(f; x_1 + (x_2 - x_1)t, y_1 + (y_2 - y_1)t) \leq z_0\}.$$

Then by the lower semicontinuity of $I(f)$ we get $S(f; x_0, y_0) \geq I(f; x_0, y_0)$, where $x_0 = x_1 + (x_2 - x_1)t_0$, $y_0 = y_1 + (y_2 - y_1)t_0$, and the lemma is proved.

The proof of the following lemma is similar to the proof of the previous one.

Lemma 2.2. *If $f, g \in SC_D$, and $(x_1, y_1), (x_2, y_2) \in D$ are such that*

$$I(f; x_1, y_1) > S(g; x_1, y_1) \text{ and } I(g; x_2, y_2) > S(f; x_2, y_2),$$

then there exists $t_0 \in [0, 1]$ for which

$$f(x_1 + (x_2 - x_1)t_0, y_1 + (y_2 - y_1)t_0) \cap g(x_1 + (x_2 - x_1)t_0, y_1 + (y_2 - y_1)t_0) \neq \emptyset.$$

According to the definition of the set SC_Ω, for every $f \in SC_\Omega$ there exist two functions $\varphi(x) = I(f; x)$ and $\psi(x) = S(f; x)$, where φ is lower semicontunuous and ψ is upper smicontinuous, such that $f(x) = [\varphi(x), \psi(x)]$. The converse statement also holds, as we now show.

Lemma 2.3. *If $\varphi, \psi \in B_\Omega$, where φ is lower semicontunuous and ψ is upper smicontinuous, and $\varphi(x) \leq \psi(x)$ holds for every $x \in \Omega$, then the function $f(x) = [\varphi(x), \psi(x)]$ belongs to SC_Ω.*

Proof. It is easy to note that the lower semicontinuous functions are the fixed points of the operator I in B_Ω and the upper semicontinuous functions are fixed points of the operator S in B_Ω. According to the hypotheses of the lemma, we obtain

$$\begin{aligned} F(f; x) &= [I(f; x), S(f; x)] = [I(\varphi; x), S(\psi; x)] \\ &= [\varphi(x), \psi(x)] = f(x). \end{aligned}$$

This completes the proof.

3.3 Hausdorff continuous functions

Definition 2.3. Denote by HC_Ω the set of those functions $f \in SC_\Omega$ for which the conditions $g \in SC_\Omega$ and $g(x) \subset f(x)$ for all $x \in \Omega$ jointly imply $f = g$. A function $f \in HC_\Omega$ is called Hausdorff continuous (H-continuous).

The name "Hausdorff continuous" will be justified later on when we suply a characterization of H-continuity in terms of Hausdorff distance.

Every continuous function is H-continuous, but not vice versa.

Lemma 2.4. *A necessary and sufficient condition for a function $f \in B_\Omega$ to be H-continuous is*

$$S(I(f)) = S(f) \text{ and } I(S(f)) = I(f). \tag{3.3}$$

Proof. For the necessity, if (2.3) fails then

$$g(x) = [I(S(f;x)), S(I(f;x))]$$

would be properly contained in $f(x) = [I(f;x), S(f;x)]$ for some x'. Thus, for this g, we would have $g \in SC_\Omega$, $g(x) \subset f(x)$ for all $x \in \Omega$, but not $f = g$.

For sufficiency, suppose (2.3) holds and $F(g) \subset F(f)$, i.e.,

$$I(f;x) \leq I(g;x) \leq S(g;x) \leq S(f;x) \tag{3.4}$$

for all $x \in \Omega$. According to the monotonicity of the operators S and I, we obtain from (2.3) and (2.4) that $S(I(g)) = S(f)$ and $I(S(g)) = I(f)$. On the other hand, from the definition of the Baire functions, it follows that

$$S(I(g);x) \leq S(g;x), \quad I(S(g);x) \geq I(g;x),$$

and therefore $S(g) = S(f)$ and $I(g) = S(f)$, i.e., $F(g) = F(f)$.

Corollary 2.1. *The equality $F(S(f)) = F(I(f))$ is a necessary and sufficient condition for f to be H-continuous.*

Corollary 2.2. *For every function $f \in HC_\Omega$ the corresponding normal function $\hat{f} \in B_\Omega$ is such that $\hat{f} \subset f$ and $F(\hat{f}) = F(f) = f$, or $f \approx \hat{f}$. This shows that the completed graph of $\hat{f}$ is f.*

We define the modified extended Baire functions for every segment function $f \in BS_\Omega$ and hence for every function $f \in B_\Omega$

$$I_0(\delta, f; x) = \inf\{z : z \in f(u);\ u \in O_0(x,\delta) \cap \Omega\},$$

$$S_0(\delta, f; x) = \sup\{z : z \in f(u);\ u \in O_0(x,\delta) \cap \Omega\},$$

where $O_0(u,\delta) = \{(x,y) : x \neq u, \rho(x,u) < \delta\}$, and the modified Baire functions

$$I_0(f;x) = \lim_{\delta\to\infty} I_0(\delta, f; x), \quad S_0(f;x) = \lim_{\delta\to\infty} S_0(\delta, f; x).$$

Lemma 2.5. *A necessary condition for f to be H-continuous is: for every $x \in \Omega$ and every $z \in f(x)$*

$$I_0(f;x) \le z \le S_0(f;x),$$

and hence

$$I(f;x) = I_0(f;x) \text{ and } S(f;x) = I_0(f;x).$$

Proof. Let $f \in HC_\Omega$ and the hypotheses of the Lemma be violated at a certain point $x_1 \in \Omega$. We define a function $g \in HC_\Omega$ as follows. For $x \neq x_1$, set $g(x) = f(x)$, and

$$g(x_1) = \frac{1}{2}[I_0(f;x_1) + S_0(f;x_1)].$$

Then $F(g) \subset F(f)$ and $F(g) \neq F(f)$.

Corollary 2.3. *Let $f \in IM_\Omega$. For every point x_0 there exist a sequence $\{x'_n\}_1^\infty$, $x'_n \in \Omega$ and $x'_n \neq x_0$, such that*

$$\lim_{n\to\infty} f(x'_n) = S(f;x_0)$$

and a sequence $\{x''_n\}_1^\infty$, $x''_n \in \Omega$ and $x''_n \neq x_0$ such that

$$\lim_{n\to\infty} f(x''_n) = I(f;x_0).$$

4. Functional spaces with Hausdorff metric

Let (M,ρ) be a metric space, where $\rho(a,b)$ is the distance between two elements $a,b \in M$. In his famous book, F. Hausdorff [4] defined the distance between the subsets of a given metric space in the following way. Let $A \subset M$. We shall denote by $U(\epsilon, A)$, where $\epsilon \ge 0$, the set of all points $x \in M$ such that $\rho(x;A) \le \epsilon$, i.e.,

$$U(\epsilon, A) = \{x : x \in M \text{ and } \rho(x;A) \le \epsilon\}.$$

The infimum $h(A,B)$ of those ϵ for which $U(\epsilon,A) \supset B$ and $U(\epsilon,B) \supset A$ is called the Hausdorff distance or distance in the sense of deviation of sets between the sets A and B induced by the distance ρ. If M is complete and 2^M is the set of all compact subsets of M then the Hausdorff distance between subsets is a metric in 2^M.

4.1 Hausdorff distance between functions

We shall define the Hausdorff distance in functional classes. Let for $\alpha > 0$

(4.5) $$\rho_\alpha((x,z),(\xi,\zeta)) = \max\{|z-\zeta|, \alpha^{-1}\rho(x,\xi)\}$$

be the box distance in $\Omega\times\mathbf{R}$.

Definition 3.1. The Hausdorff distance with parameter α [10] between two functions $f, g \in BS_\Omega$ is defined as the Hausdorff distantce between the two closed and bounded point sets $F(f)$, $F(g)$ in $\Omega\times\mathbf{R}$. This distance shall be denoted by $r(\alpha; f, g)$.

The Hausdorff distance between two functions from BS_Ω, according to the Definition 3.1 is defined by the formula

(4.6) $$r(\Omega,\alpha; f,g) = \max\Big\{ \sup_{(x,z)\in F(f)} \inf_{(\xi,\zeta)\in F(g)} \rho_\alpha((x,z),(\xi,\zeta)),$$
$$\sup_{(x,z)\in F(g)} \inf_{(\xi,\zeta)\in F(f)} \rho_\alpha((x,z),(\xi,\zeta))\Big\}.$$

When it is clear in which compact Ω the Hausdorff metric is defined, we will use for short $r(\alpha; f,g) = r(\Omega,\alpha; f,g)$, and $r(f,g)$ if $\alpha = 1$.

Now we shall define the Hausdorff distance in another way only for the functions from HC_Ω and shall prove the equivalence of the two definitions.

Definition 3.2. The absolute value of the $\ominus$ – difference with parameter $\alpha > 0$ between two functions $f, g \in B_\Omega$ is

$$|f(x)\ominus_\alpha g(x)| = \max\Big\{ \inf_{(\xi,\zeta)\in F(g)} \max[|f(x)-\zeta|, \alpha^{-1}\rho(x,\xi)],$$
$$\inf_{(\xi,\zeta)\in F(f)} \max[|g(x)-\zeta|, \alpha^{-1}\rho(x,\xi)]\Big\}.$$

Theorem 3.1. *If $f, g \in HC_\Omega$, then*

$$r(\alpha; f,g) = \sup_{x\in\Omega} |f(x)\ominus_\alpha g(x)|.$$

Proof. As the completed graphs of the functions are convex toward the z axis, in (3.6) we may replace under two max'es

$$(x,z)\in F(g) \text{ by } x\in\Omega, : z = I(g;x), S(g;x)$$

and

$$(x,z)\in F(f) \text{ by } x\in\Omega, : z = I(f;x), S(f;x).$$

Then the Theorem follows from Corollary 2.3.

The next four lemmas follow directly from the definition of the Hausdorff distance between functions, when Ω is linear space and the distance ρ in Ω is homogenous, i.e., if $\lambda \geq 0$ is a constant, then for every $x, \xi \in \Omega$ we have $\rho(\lambda x, \lambda\xi) = \lambda\rho(x,\xi)$.

Lemma 3.1. *Let the compact Ω be a linear space, the distance in Ω be homogeneous, $f, g \in B_\Omega$ and c is a constant, then*

$$r(\alpha; f+c, g+c) = r(\alpha; f, g),$$

$$r(\alpha; cf, cg) = |c| r(\alpha|c|; f, g),$$

$$r(\alpha; f, g) \leq \max\{1, \alpha'/\alpha\} r(\alpha'; f, g).$$

Lemma 3.2. *If $f, g, \varphi, \psi \in B_\Omega$ and for every $x \in \Omega$ the inequalities*

$$\varphi(x) \leq f(x) \leq \psi(x),$$

$$\varphi(x) \leq g(x) \leq \psi(x),$$

hold, then

$$r(\alpha; f, g) \leq r(\alpha, \varphi, \psi).$$

Lemma 3.3. *Let $f, g \in B_\Omega$ and $\Omega_\delta \subset \Omega$ be such that for every $x_0 \in \Omega_\delta$ there exists a $x \in \Omega$ with $\rho(x_0, x) < \delta$. If $f(x) = g(x)$ for $x \notin \Omega_\delta$, then $r(\alpha; f, g) \leq \alpha^{-1}\delta$.*

Lemma 3.4. *For every four tunctions $f_1, f_2, g_1, g_2 \in B_\Omega$ we have*

$$r(\alpha; f_1 + f_2, g_1 + g_2) \leq r(\alpha; f_1, g_1) + r(\alpha; f_2, g_2),$$

and hence

$$r\Big(\sum_{n=1}^{N} f_n, \sum_{n=1}^{N} g_n\Big) \leq \sum_{n=1}^{N} r(\alpha; f_n, g_n).$$

Definition 3.3. Let Ω be a compact and $\check{\Omega} = \Omega \setminus \partial\Omega$ be its interior. We say that the compact Ω is t i l e d by the compacts $\{\Omega_n\}_1^N$ if $\Omega = \bigcup_1^N \Omega_n$ and $\check{\Omega}_i \bigcup \check{\Omega}_j = \emptyset$ for $i \neq j$.

A division of the square D in smaller (closed) squares $\{D_i\}_1^{n^2}$, by lines parallel to the coordinate axes, is an example for tiling of D.

Lemma 3.5. (First Tiling Lemma). *Let $\Omega = \bigcup_1^N \Omega_n$ be a tiling of the compact Ω, $f, g \in SC_\Omega$ and let $f_n, g_n \in SC^\alpha_{\Omega_n}$; $1 \leq nleqN$ be the restrictions of f, g on Ω_n respectively, i.e.,*

$$f_n(x) = \begin{cases} f(x) & \text{for} x \in \Omega_n, \\ 0 & \text{for} x \in \Omega \setminus \Omega_n, \end{cases}, \quad 1 \leq n \leq N.$$

Then:

$$r(\Omega, \alpha; f, g) \leq \max_{1 \leq n \leq N} r(\Omega_n, \alpha; f_n, g_n).$$

P r o o f. We use the definition (3.6) of the Hausdorff distance. Obviously, for every $x \in \Omega_n$ and$z \in F(f)$ we have

$$\begin{aligned} A(x,z) &= \inf_{\xi \in \Omega,\ \zeta \in F(g;\xi)} \max[|z-\zeta|, \alpha^{-1}\rho(x,\xi)] \\ &\leq \inf_{\xi \in \Omega_n,\ \zeta \in F(g;\xi)} \max[|z-\zeta|, \alpha^{-1}\rho(x,\xi)] = A_n(x,z), \end{aligned}$$

and for every $x \in \Omega_n$ and $z \in F(g)$ we have

$$\begin{aligned} B(x,z) &= \inf_{\xi \in \Omega,\ \zeta \in F(f;\xi)} \max[|z-\zeta|, \alpha^{-1}\rho(x,\xi)] \\ &\leq \inf_{\xi \in \Omega_n,\ \zeta \in F(f;\xi)} \max[|z-\zeta|, \alpha^{-1}\rho(x,\xi)] = B_n(x,z), \end{aligned}$$

as $\Omega_n \subset \Omega$, since

$$\begin{aligned} r(\Omega,\alpha;f,g) &= \max\{\sup_{x\in\Omega, z\in F(f)} A(x,z), \sup_{x\in\Omega, z\in F(g)} B(x,z)\} \\ &\leq \max\{\sup_{x\in\Omega_n, z\in F(f)} A_n(x,z), \sup_{x\in\Omega_n, z\in F(g)} B_n(x,z) = r(\Omega_n,\alpha;f_n,g_n); \\ &\qquad\qquad 1 \leq n \leq N, \end{aligned}$$

which completes the proof.

4.2 The space of the segment continuous functions is complete

We now focus our attention to the completeness of the metric space $(SC_D, r(\alpha)) = SC_D^\alpha$. In order to prove this completeness, we need some auxiliary statements that will be proved in advance.[3]

Let $\Delta = \{(x,y) : x_0 \leq x \leq x_0 + 4\delta,\ y_0 \leq y \leq y_0 + 4\delta\}$ be a square divided in 16 subsquares $\Delta_{i,j} = \{(x,y) : x_0 + i\delta \leq x \leq x_0 + (i+1)\delta,\ y_0 + j\delta \leq y \leq y_0 + (j+1)\delta\}$; $i,j = 0,1,2,3$. For every four values $M_{i,j}$; $i,j = 0,1$, attached to the corners on Δ, we can uniquely define a continuous function χ on Δ interpolating the values $M_{i,j}$ if:

i)χ is constant on the corner subsquares,

ii)χ is a linear function of one variable on the border subsquares, and

iii)χ is a quadratic function of the form $axy + bx + cy + d$ on the inner subsquares.

---

[3] The same method may be used to prove the completeness of SC_Ω if Ω is a cube in the m-dimentional Euclidean space R_3. The case $m = 1$ was proved in [9].

Precisely, the function $\chi(x,y) = \chi(\Delta; M_{0,0}, M_{1,0}, M_{0,1}, M_{1,1}; x, y)$ is defined as follows:

$\chi(x,y) = M_{i,j}$ on $\Delta_{3i,3j}$; $i,j = 0,1$,
$\chi(x,y) = 2\delta^{-1}(M_{1,j} - M_{0,j})(x - x_0 - \delta) + M_{0,j}$ on $\Delta_{1,3j} \cup \Delta_{2,3j}$; $j = 0,1$,
$\chi(x,y) = 2\delta^{-1}(M_{i,1} - M_{i,0})(y - y_0 - \delta) + M_{i,0}$ on $\Delta_{3i,1} \cup \Delta_{3i,2}$; $i = 0,1$,
$\chi(x,y) = 4\delta^{-2}(M_{1,1} - M_{1,0} - M_{0,1} + M_{0,0})(x - x_0 - \delta)(y - y_0 - \delta) +$
$2\delta^{-1}(M_{1,0} - M_{0,0})(x - x_0 - \delta) + 2\delta^{-1}(M_{0,1} - M_{0,0})(y - y_0 - \delta) + M_{0,0}$
on $\cup\Delta_{i,j}$; $i,j = 1,2$.

From the explicit definition of χ it is easy to estimate the modulus of continuity of this function.

$$\omega(\chi;t) \le 32M\delta^{-1}t \quad \text{where} \quad M = \max\{|M_{i,j}| : i,j = 0,1\}. \tag{4.7}$$

Lemma 3.6. *For every function $f \in BS_D$ and for every $\delta > 0$ there exists a continuous function $\psi \in C_D$ such that*

$$S(\delta, f; x, y) \le \psi(x,y) \le S(4\delta, f; x, y) \tag{4.8}$$

for all (x,y) in D and such that the modulus of continuity of ψ satisfies the inequality

$$\omega(\psi;t) \le 32\|f\|\delta^{-1}t. \tag{4.9}$$

Proof. Let $\delta > 0$, $x_i, y_i = 4i\delta$ and $\overline{\Delta}_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ for $i,j = 1,2,3,\ldots,N$, where $N = 1/4\delta$.(Without loss of generality, we may consider that $1/4\delta$ is a natural number.) Let

$$M'_{i,j} = \sup\{z : z \in f(x,y),\ (x,y) \in \Delta_{i,j}\}$$

and

$$M_{i,j} = \max\{M'_{k,l} : i - k + j - l \le 1,\ k \le i,\ l \le j\}.$$

We define

$$\psi(x,y) = \chi(\overline{\Delta}_{i,j}; M_{i-1,j-1}, M_{i,j-1}, M_{i-1,j}, M_{i,j}; x, y) \quad \text{fot}(x,y) \in \overline{\Delta}_{i,j}.$$

It is obvious that ψ is continuous and satisfies the inequality (3.9).
The validity of the inequalities (3.8) is directly checked.

The proof of the next Lemma is the same.

Lemma 3.7. *For every function $f \in BS_D$ and for every $\delta > 0$ there exists a continuous function $\varphi \in C_D$ such that*

$$S(\delta, f; x, y) \leq \varphi(x, y) \leq S(4\delta, f; x, y) \tag{4.10}$$

for all (x, y) in D and such that the modulus of continuity of φ satisfies the inequality $\omega(\varphi; t) \leq 32\|f\|\delta^{-1}t$.

Now we shall prove the completeness of the metric space SC_D^α. Let us mention, that if we consider the completed graphs of the elements of SC_D^α as closed end bounded point sets in R_3, metrized by the Hausdorff metric $r(\alpha)$, every Cauchy sequence of functions in SC_D^α has a limit point set, which is closed and bounded. But we have to prove that this point set is a completed graph of a function from SC_D^α.

Theorem 3.2. *The space SC_D^α is a complete metric space.*

Proof. Let $\{\hat{f}_n\}_1^\infty$ be a Cauchy sequence of functions in SC_D. Then for every $\epsilon > 0$ there exists $n(\epsilon)$ such that for $p, q > n(\epsilon)$ the inequality

$$r(\alpha; f_p, f_q) < \epsilon \tag{4.11}$$

holds. It is necessary to prove that there exists $f_0 \in SC_D$ and

$$\lim_{n\to\infty} r(\alpha; f_n, f_0) = 0. \tag{4.12}$$

From the definition of the Hausdorff distance follows that every Cauchy sequence is uniformly bounded and, i.e., there exists a number $M > 0$ such that

$$\|f_n\| < M \quad \text{for } n = 1, 2, 3, \ldots.$$

For every f_n and $\delta > 0$ we choose a function $\psi_{n,\delta}$ as in Lemma 3.6. All the functions of the sequence $\{\psi_{n,\delta}\}_1^\infty$ have modulus of continuity $\omega(\psi_{n,\delta}; t) \leq 32M\delta^{-1}t$ and therefore are equicontinuous. Consequently the function ψ_δ defined by

$$\psi_\delta(x, y) = \underline{\lim}_{n\to\infty} \psi_{n,\delta}(x, y)$$

has the same modulus of continuity. The same can be said for

$$\varphi_\delta(x, y) = \overline{\lim_{n\to\infty}} \varphi_{n,\delta}(x, y)$$

where the function $\varphi_{n,\delta}$ is chosen for f_n as in Lemma 3.7. From the way we have defined the functions $\psi_{n,\delta}$ it follows that

$$\psi_{n,2\delta}(x, y) \geq \psi_{n,\delta}(x, y) \quad \text{and} \quad \varphi_{n,2\delta} \leq \varphi_{n,\delta}(x, y),$$

and hence we have

$$\varphi_{n,2\delta}(x,y) \le \varphi_{n,\delta}(x,y) \le \psi_{n,\delta}(x,y) \le \psi_{n,2\delta}(x,y). \tag{4.13}$$

Then for $\delta = 1/2, 1/4, 1/8, \dots$ we have two sequences of functions $\{\varphi_n\}_1^\infty$ and $\{\psi_n\}_1^\infty$ such that

$$\varphi_n(x,y) \le \varphi_{n+1}(x,y) \le \psi_n(x,y) \le \psi_{n+1}(x,y)$$

and hence, if we write

$$\varphi(x,y) = \lim_{n\to\infty} \varphi_n(x,y), \quad \psi(x,y) = \lim_{n\to\infty} \psi_n(x,y), \tag{4.14}$$

then φ is lower semicontinuous and ψ is upper semicontinuous. Thus, by Lemma 2.3, the function

$$f_0(x,y) = [\varphi(x,y), \psi(x,y)] \tag{4.15}$$

belongs to SC_D.

Let us now show that (3.12) holds for the function (3.15). It is sufficient to establish for every $\epsilon > 0$ the existance of $n_0 = n_0(\epsilon)$ such that for $n > n_0$, we have

$$r(\alpha; f_n, f_0) < \epsilon. \tag{4.16}$$

Since the functions $\psi_{n,\delta}$ are uniformely continuous (because $\omega(\psi_{n,\delta}; t) \le 32\delta^{-1}Mt$), for each $\epsilon > 0$ and $(x,y) \in D$ there exists $n_0 = n_0(\epsilon)$ such that for all $(x,y) \in D$ and $n > n_0$ we have

$$|\psi_{n,\delta}(x,y) - \psi_\delta(x,y)| < \epsilon. \tag{4.17}$$

According to Lemma 3.6,

$$S(\delta, f_n; x, y) \le \psi_{n,\delta}(x,y) \le S(4\delta, f_n; x, y). \tag{4.18}$$

We obtain from (3.17) and (3.18) that

$$S(\delta, f_n; x, y) - \epsilon \le \psi_\delta(x,y) \le S(4\delta, f_n; x, y) + \epsilon. \tag{4.19}$$

The left-side inequality of (3.19) gives $S(f_n; x, y) - \epsilon \le \psi_\delta(x,y)$ and according to (3.14) and (3.15) we have

$$S(f_0; x, y \ge S(f_n; x, y) - \epsilon. \tag{4.20}$$

The right-side inequality of (3.19) according to (3.13) gives

$$S(f_0; x, y) \le S(4\delta, f_n; x, y) + \epsilon. \tag{4.21}$$

We can prove in similar way that

$$I(4\delta, f_n; x, y) - \epsilon \le I(f_0; x, y) \le I(f_n; x, y) + \epsilon. \tag{4.22}$$

From (3.10) – (3.22) and $\delta = \alpha\epsilon/4$ it follows that

$$I(\alpha\epsilon, f_n; x, y) - \epsilon \le I(f_0; x, y) \le S(f_0; x, y) \le S(\alpha\epsilon, f_n; x, y) + \epsilon, \tag{4.23}$$

$$I(f_0; x, y) - \epsilon \le I(f_n; x, y) \le S(f_n; x, y) \le S(f_o; x, y) + \epsilon. \tag{4.24}$$

The equation (3.23) shows that for every $(x, y) \in D$ and for every $z \in F(f_0; x, y)$ there exist $(\xi, \eta) \in [x - \alpha\epsilon, x + \alpha\epsilon] \times [y - \alpha\epsilon, y + \alpha\epsilon] \cap D$ and $\zeta \in F(f_n; \xi, \eta)$ such that $|z - \zeta| < \epsilon$. The inequality (3.16) provides that for every $(x, y) \in D$ and every $z \in F(f_n, x, y)$ there exists $\zeta \in F(f_0, x, y)$ such that $|z - \zeta| < \epsilon$. Applying Lemma 2.2, we obtain (3.16). The theorem is proved.

From the proof of the last theorem, (3.20) and Lemma 3.2 we have.

Corollary 3.1. *If $\{f_n\}_1^\infty$ is a Cauchy sequence in SC_D^α and f_0 is the limit of this sequence, then for every $\epsilon > 0$ there exists $n_0 = n_0(\epsilon)$ such that for every $n > n_0$, the equality*

$$r(\alpha; I(f_0), S(f_0)) \le r(\alpha; I(\epsilon, f_n), S(\epsilon, f_n)) + \epsilon$$

is valid.

Let us mention that the space HC_D of the H-continuous functions is not complete under the Hausdorff metric, on the analogy of the incompleteness of the set of continuous functions under the uniform norm.

4.3 Modulus of H-continuity

Definition 3.4. Let $f \in B_\Omega$. We define the modulus of H-continuity (with paramerer α) of f by the formula

$$\tau(\alpha, f; \delta) = r(\alpha; S(\delta/2, f), I(\delta/2, f)).$$

Directly from the definition of the modulus of H-continuity one obtains the following facts:

i) The modulus of H-continuity does not exceed the modulus of continuity

$$\tau(\alpha, f; \delta) \le \omega(f; \delta). \tag{4.25}$$

Indeed, by definition

$$\tau(\alpha, f; \delta) = r(\alpha; S(\delta/2, f), I(\delta/2, f)) \leq \sup_{x \in D} |S(\delta/2, f; x) - I(\delta/2, f; x)|$$
$$= \sup_{x_1, x_2 \in \Omega,:\ \rho(x_1, x_2) \leq \delta} \{|z - \zeta| :\ z \in F(f; x_1);\ \zeta \in F(f; x_2)\} = \omega(f; \delta).$$

ii) The modulus of H-continuity tends to the modulus of continuity as the parameter tends to zero:

$$\lim_{\alpha \to +0} \tau(\alpha, f; \delta) = \omega(f; \delta).$$

iii) The modulus of H-continuity is a monotone nondecreasing function of δ, i.e., if $\delta_1 \leq \delta_2$, then $\tau(\alpha, f; \delta_1) \leq \tau(\alpha, f; \delta_2)$.

Lemma 3.8. (Second Tiling Lemma). *Let $\Omega = \bigcup_1^N \Omega_n$ be a tiling of the compact Ω, $f \in SC_\Omega$ and let $f_n \in SC^\alpha_{\Omega_n}$; $1 \leq n \leq N$ be the restrictions of f on Ω_n, i.e.,*

$$f_n(x) = \begin{cases} f(x) & \text{for} x \in \Omega_n, \\ 0 & \text{for} x \in \Omega \setminus \Omega_n, \end{cases} \quad 1 \leq n \leq N.$$

Then:

$$\tau(\Omega, \alpha; f, g) \leq \alpha^{-1}\delta + \max_{1 \leq n \leq N} \tau(\Omega_n, \alpha; f_n, g_n).$$

Proof. Let us define the functions

$$\underline{f}(\delta; x) = I(\Omega, \delta/2, f; x), \text{ and } \overline{f}(\delta; x) = S(\Omega, \delta/2, f; x).$$

According to the Definition 3.4 and Lemma 3.5 we have

$$(4.26) \qquad \tau(\Omega, \alpha, f; \delta) = r(\Omega, \alpha; \underline{f}, \overline{f})) \leq \max_{1 \leq n \leq N} r(\Omega_n, \alpha; \underline{f}_n), \overline{f}_n)).$$

Let $\Omega_{n,\delta} = \{x :\ x \in \Omega_n \text{ and } \rho(x, \partial\Omega_n) < \delta\}$, where, as usual, $\rho(x, A) = \inf_{\xi \in A} \rho(x.\xi)$. By the definition,

$$I(\Omega_n, \delta/2, f_n; x) = \underline{f}(\delta; x) \text{ and } S(\Omega_n, \delta/2, f_n; x) = \overline{f}(\delta, x)$$
$$\text{for } x \in \Omega \setminus \Omega_{n,\delta/2};\ 1 \leq n \leq N.$$

Then, using Lemma 3.3, we obtain

$$r(\Omega_n, \alpha; \underline{f}_n(\delta), \overline{f}_n(\delta)) \leq r(\Omega_n, \alpha; \underline{f}_n(\delta), I(\Omega_n, \delta/2, f_n; x)) +$$
$$r(\Omega_n, \alpha; I(\Omega_n, \delta/2, f_n; x), S(\Omega_n, \delta/2, f_n; x)) +$$
$$r(\Omega_n, \alpha; S(\Omega_n, \delta/2, f_n; x), \overline{f}_n(\delta)) \leq$$
$$\alpha^{-1}\delta/2 + \tau(\Omega_n, \alpha, f_n; \delta) + \alpha^{-1}\delta/2,$$

and according to (3.26), the proof is completed.

We shall give now definition of H-continuous function through the modulus of H-continuity, which justified the name Hausdorff continuity.

Theorem 3.3. *The function $f \in B_D$ is H-continuous if and only if*

$$\lim_{\delta \to +0} \tau(\alpha, f; \delta) = \tau(\alpha, f; 0) = 0.$$

Proof. According to the definition of the modulus of H-continuity, it follows that

$$\lim_{\delta \to +0} \tau(\alpha, f; \delta) = \lim_{\delta \to +0} r(\alpha; S(\delta/2, f), I(\delta/2, f))$$

$$= r(\alpha; S(f), I(f)) = r(\alpha; F(S(f)), F(I(f))).$$

We see that $\lim_{\delta \to +0} \tau(\alpha, f; \delta) = 0$ holds if and only if $F(S(f)) = F(I(f))$. According to Corollary 2.1, this is a necessary and sufficient condition for f to be H-continuous. The Theorem is proved.

Definition 3.5. *The set of function $HC_{\Omega,\tau} \subset HC_\Omega$ is called H-equicontinuous if there exists a nondecreasing function τ with*

$$\lim_{\delta \to +0} \tau(\delta) = 0,$$

such that for every $f \in HC_{\Omega,\tau}$ the inequality

$$\tau(\alpha, f; \delta) \leq \tau(\delta); \quad \delta > 0$$

is valid.

Theorem 3.4. *The set of H-equicontinuous functions $HC_{D,\tau}$ is complete.*

Proof. Let $\{f_n\}_1^\infty$ be a Cauchy sequence in $HC_{D,\tau} \subset SC_D^\alpha$ and f_0 be the limit of this sequence in SC_D^α, which exists according to Theorem 3.2. Then, according to the hypothesis of the Theorem and the Corollary 3.1, for every $\epsilon > 0$ there exists $n_0 = n_0(\epsilon)$ such that for every $n > n_0$, the equalities

$$\begin{aligned} \tau(\alpha, f_0; 0) &= r(\alpha; I(f_0), S(f_0)) \leq r(\alpha; I(\epsilon, f_n), S(\epsilon, f_n)) + \epsilon \\ &= \tau(\alpha, f_n; 2\epsilon) + \epsilon \leq \tau(2\epsilon) + \epsilon, \end{aligned}$$

hold. As the number $\epsilon > 0$ is arbitrarily chosen, since $\tau(\alpha, f_0; 0) = 0$ and the Theorem is proved.

5. The space of image functions

Having in mind the eight properties formulated in [2] an the additional properties given in the Introduction we define an image modeling space of functions.

Definition 4.1. The set of the H-continuous functions defined on the compact Ω and normalized with the condition[4]

$$f(x) = S(f;x) \quad \textit{for every} \quad x \in \Omega, \tag{5.27}$$

is denoted by IM_Ω. We call a function $f \in IM_\Omega$ an image function with support Ω.

The metric space $(IM_\Omega, r(\alpha))$, where $r(\alpha) = r(\alpha; f, g)$ is the Hausdorff distance with parameter α, is denoted by IM_Ω^α. We call the space IM_Ω^α an image space.

The space of the functions $f \in IM_\Omega^\alpha$ with modulus of H-continuity $\tau(\Omega, \alpha, f; \delta)$ satisfying the condition

$$\tau(\Omega, \alpha, f; \delta) \leq \alpha^{-1}\nu\delta \tag{5.28}$$

is denoted by $IM_\Omega^{\alpha,\nu}$. We call the space $IM_\Omega^{\alpha,\nu}$ an image space of class ν; and a function $f \in IM_\Omega^{\alpha,\nu}$ a image function of class ν.

We will be interested mostly in the case $\Omega = D$ and shall show that the *fractal transform operators* [2][p. 186] are operating only in image spaces of class $\nu \geq 1$.

We claim that the functions from $IM_D^{\alpha,\nu}$, for $\nu \geq 1$, are the natural mathematical models for the grayscale images. Some arguments for this follow.

Theorem 4.1. *Every image function is Lebesgue integrable, and every image space IM_D^α of finite class is complete.*

The proof of this theorem follows from the upper semicontinuity of the image functions, according to (4.27), and from Theorem 3.4.

Definition 4.2. Let $\Omega' \subset \Omega$ be a compact. Following [2] [p.11], we define a clipping operation $c(\Omega', f) : : IM_\Omega \to IM_\Omega$ as follows:

$$c(\Omega', f; x) = \begin{cases} f(x) & \text{for} x \in \Omega', \\ 0 & \text{for} x \in \Omega \setminus \Omega'. \end{cases}$$

It is immediately seen that every image space IM_Ω^α is closed under the operation clipping. From the Second Tiling Lemma 3.8 it follows that the space $IM_\Omega^{\alpha,\nu}$ is closed under the operation clipping only if it is of the class $\nu \geq 1$.

If $\Omega' = \Omega$, then $c(\Omega, f; x) = f(x)$.

The image space IM_D is a linear space.

[4] It is natural to have an additional condition $0 \leq f(x) \leq 1$.

Definition 4.3. For two functions $f, g \in IM_D$ we define the function $h \in IM_D$ as the sum of f and g in IM_D as follows. We take the normal functions $\hat{f}, \hat{g}$, corresponding to f, g, and the normal function $\hat{h} = \hat{f} + \hat{g}$. Then h is the uniquely determined function from IM_D equivalent to $\hat{h}$.

If $f \in IM_D$ and λ is a number, then the function λf is defined as the function $g \in IM_D$ equivalent to the function $\lambda\hat{f}$.

In short, the Definition 4.3 tells us that after every arithmetic operation with functions from IM_D we have to replace the result with the equivalent function from IM_D. This is natural, as we do not consider two functions different if they have equal completed graphs. In this way we mav consider IM_D as a linear space.

5.1 Operators in the space of the image functions

Definition 4.4. An operator $\Theta : M \to M$ in the metric space $R = (M, r)$ is a contractive operator with contractivity factor $\lambda < 1$ if for every pair of functions $f, g \in M$ the inequality

$$r(\Theta(f), \Theta(g)) \leq \lambda r(f, g)$$

holds.

The following theorem is a common knowledge.

Theorem 4.2. *If $R = (M, r)$ is a complete metric space and Θ is a contractive operator in R with contractivity factor $\lambda < 1$, then:*

i) There exists a unique eigen-function (fixed point) f_Θ satisfying the equation $\Theta(f) = f$.

ii) For every $f_0 \in R$, the iterative sequence $f_m = \Theta(f_{m-1})$; $m = 1, 2, 3, \ldots$ is convergent and has as a limit the fixed point f_Θ.

iii) There is an error estimation

$$r(f_\Theta, f_m) \leq \frac{\lambda^m}{1-\lambda} r(f_0, f_1).$$

We shall define an useful contractive operator.

Definition 4.5. Let the function $v : \mathbf{R} \to \mathbf{R}$ is Lipshitz with Lipshitz constant $\lambda > 0$, i.e.,

$$|v(t) - v(t')| \leq \lambda |t - t'|.$$

If $\lambda < 1$, we call v contractive function with contractivity factor λ. The set of this functions is denoted by L_λ.

Let $w : \Omega \to \Omega$ be an invertible Lipshitz tansformation with Lipshitz constant λ, i.e.,

$$\rho(w(x), w(\xi)) \leq \lambda \rho(x, \xi).$$

and $w^{-1}(w(x)) = x$. If $\lambda < 1$, we call w contractive mapping with contractivity factor λ. The set of these mapings is denoted by $L_{\Omega,\lambda}$.

Definition 4.6. Let $v \in L_\lambda$, $w \in L_{\Omega,\lambda}$; $\lambda < 1$ and $\Omega' = w(\Omega) = \{x : x = w(\xi), \xi \in \Omega\}$. We define the operator $\Lambda(v, w)(f) : IM_\Omega \to IM_\Omega$ in the following way:

$$\Lambda(v,w)(f;x) = \begin{cases} v(f(w^{-1}(x))) & \text{if} x \in \breve{\Omega}', \\ 0 & \text{if} x \in \Omega \setminus \Omega', \\ \text{as element of } IM_\Omega & \text{if} x \in \partial\Omega'. \end{cases} ,$$

where $\partial\Omega$ is the contour of Ω and $\breve{\Omega} = \Omega \setminus \partial\Omega$ is its interior.

Lemma 4.1. *The operator $\Lambda(v, w)$, determined by Definition 4.6 is a contractive operator from IM_Ω in IM_Ω with contractivity factor λ, i.e., for every two functions $f, g \in IM_\Omega$ we have*

$$r(\Omega, \alpha; \Lambda(v, w)(f), \Lambda(v, w)(g)) \leq \lambda r(\Omega, \alpha; f, g). \tag{5.29}$$

Proof. According to First tiling lemma,

$$r(\Omega, \alpha; \Lambda(v, w)(f), \Lambda(v, w)(g)) = r(\Omega', \alpha; \Lambda(v, w)(f), \Lambda(v, w)(g)),$$

as

$$r(\overline{\Omega \setminus \Omega'}, \alpha; \Lambda(v, w)(f), \Lambda(v, w)(g)) = 0.$$

First we estimate

$$\begin{aligned} &\inf_{\xi\in\Omega',\, \zeta\in F(\Lambda(g);\xi)} \max[|\Lambda(f;x) - \zeta|, \alpha^{-1}\rho(x,\xi)] \\ &= \inf_{\xi\in\Omega',\, \zeta\in F(\Lambda(g);\xi)} \max[|v(f(w^{-1}(x))) - v(w^{-1}(\zeta))|, \alpha^{-1}\rho(x,\xi)] \\ &\leq \inf_{\xi\in\Omega,\, \zeta\in F(g;\xi)} \max[|v(f(x)) - v(\zeta)|, \alpha^{-1}\rho(w(x), w(\xi))] \\ &\leq \inf_{\xi\in\Omega,\, \zeta\in F(g;\xi)} \max[\lambda[|f(x) - \zeta|, \lambda\alpha^{-1}\rho(x,\xi)] \\ &\leq \lambda \inf_{\xi\in\Omega,\, \zeta\in F(g;\xi)} \max[|f(x) - \zeta|, \alpha^{-1}\rho(x,\xi)], \end{aligned}$$

and in the same way

$$\inf_{\xi\in\Omega',\,\zeta\in F(\Lambda(f);\xi)} \max[|\Lambda(g;x)-\zeta|,\alpha^{-1}\rho(x,\xi)]$$
$$\leq \lambda \inf_{\xi\in\Omega,\,\zeta\in F(f;\xi)} \max[|g(x)-\zeta|,\alpha^{-1}\rho(x,\xi)],$$

which proves (4.29).

From (4.29) we have that

$$r(\Omega,\alpha;I(\Lambda(f)),S(\Lambda(f)))\leq \lambda r(\Omega,\alpha;I(f),S(f))=0,$$

as $f\in IM_\Omega$, since $\Lambda(v,w)(f)\in IM_\Omega$, which completes the proof.

Corollary 4.1. *Let $\Lambda(v,w)$ is the contractive operator with contractivity factor $\lambda<1$, determined by Definition 4.6 and $p>0$ is a number, then*

(5.30) $$r(\Omega,\alpha;p\Lambda(v,pw)(f),p\Lambda(v,pw)(g))\leq p\lambda r(\Omega,\alpha;f,g).$$

Theorem 4.3. *The operator $\Lambda(v,w)$ with contractivity factor $\lambda<1$ is an operator in $IM_D^{\alpha,\nu}$, i.e., $\Lambda(v,w):IM_D^{\alpha,\nu}\to IM_D^{\alpha,\nu}$ if*

(5.31) $$\lambda\leq 1-\frac{1}{\nu}.$$

Proof. Let $f\in IM_\Omega^{\alpha,\nu}$ and $\Lambda(v,w)(f)=\Lambda(f)$. According to Second Tiling Lemma 3.8 and Lemma 4.1, if $\Omega_1=w(\Omega)$, we have

$$\tau(\Omega,\alpha,\Lambda(f);\delta)\leq\alpha^{-1}\delta+\tau(\Omega_1,\alpha,\Lambda(f);\delta)$$
$$\leq\alpha^{-1}\delta+\lambda\tau(\Omega,\alpha,f;\delta).$$

And in view of (4.31)

$$\tau(\Omega,\alpha,\Lambda(f);\delta)\leq(1+\lambda\nu)\leq\alpha^{-1}\nu\delta,$$

since $\Lambda(f)\in IM_\Omega^{\alpha,\nu}$, which completes the proof.

5.2 Iterated function systems

In this part we shall prove some analogs of the basic theorems of M. F. Barnsley and L. P. Hurd [2] [pages:186,187] for convergence of Fractal Transform in the spaces IM_D^α. Our goal here is mostly theoretical.

In Fractal Image Compression, two types of operators are used [2], which may be global or local. We shall prove that these operators are operators in the image spaces. Let us mention, that without Hausdorff distance, which we introduce, the theorems for the convergions of the iterative siquences of these operators were not satisfactory.

The notion *iterated function sistem* (IFS), is well established, see for example [6,5,7,2,3]. We will consider IFS in the complete metric space $IM_\Omega^{\alpha,\nu}$.

Definition 4.7. Let $\Lambda_1^N = \{\Lambda(v_n, w_n)\}_1^N$ be operators settled in Definition 4.5 with contractivity factors $\lambda < 1$, and let $D_n = w_n(D)$; $n = 1, 2, 3, \ldots, N$, where $\{D_n\}_1^N$ is a tiling of D. We define the operator $\Psi(\Lambda_1^N)(f;x) = \Psi(f) : IM_D^\alpha \to IM_D^\alpha$ as follows

$$\Psi(f;x) = \begin{cases} \Lambda_n(v_n, w_n)(f;x) & \text{if} x \in \check{D}_n, \\ & \\ \text{as element of } IM_D & \text{if} x \in \partial D_n \end{cases} ; \quad n = 1, 2, 3, \ldots, N.$$

The operator Ψ is called an IFS operator, or a tiling operator.

Let, in addition to the definition of the operator Ψ, $\{D'_n\}_1^N$, $D'_n \subset D$ be compacts and $C_1^N = \{c(D'_n, f)\}_1^N$ be clipping operations (Definition 4.2). We define the operator $\Xi(\Lambda_1^N,, C_1^N)(f;x) = \Xi(f) : IM_D^\alpha \to IM_D^\alpha$ as follows

$$\Xi(f;x) = \begin{cases} \Lambda_n(v_n, w_n)(c(D'_n, f;x)) & \text{if} x \in \check{D}_n, \\ & \\ \text{as element of } IM_D & \text{if} x \in \partial D_n \end{cases} ; \quad n = 1, 2, 3, \ldots, N.$$

The operator Ξ is called local IFS operator or local tiling operator.

Let $P_1^N = \{p_n\}_1^N$; $p_n > 0$ and $p_1 + p_2 + \ldots + p_N = 1$, and $D_n = p_n w_n(D)$; $n = 1, 2, 3, \ldots, N$.[5] We define the operator $\Phi(\Lambda_1^N, P_1^N)(f;x) = \Phi(f) : IM_D^\alpha \to IM_D^\alpha$ as follows

$$\Phi(f;x) = \begin{cases} \sum_{n=1}^N p_n \Lambda_n(v_n, p_n w_n)(f;x) & \text{if} x \in \bigcup_{n=1}^N \check{D}_n, \\ & \\ \text{as element of } IM_D & \text{if} x \in \bigcup_{n=1}^N \partial D_n. \end{cases}$$

The operator Φ is called an IFS operator with probabilities, or a stochastic operator.

Let, in addition to the definition of the operator Φ, $\{D'_n\}_1^N$, $D'_n \subset D$ be compacts and $C_1^N = \{c(D'_n, f)\}_1^N$ be clipping operations (Definition 4.2). We define the operator $\Upsilon(\Lambda_1^N, P_1^N, C_1^N)(f;x) = \Upsilon(f) : IM_D^\alpha \to IM_D^\alpha$ as follows

$$\Upsilon(f;x) = \begin{cases} \sum_{n=1}^N p_n \Lambda_n(v_n, p_n w_n)(c(D'_n, f;x)) & \text{if} x \in \bigcup_{n=1}^N \check{D}_n, \\ & \\ \text{as element of } IM_D & \text{if} x \in \bigcup_{n=1}^N \partial D_n. \end{cases}$$

The operator Υ is called local IFS operator with probabilities, or local stochastic operator.

If Θ is any of the operators Ψ, Ξ, Φ, Υ, then Θ is called a fractal transform operator.

We are ready now to prove the basic theorem for the grayscale fractal transform operators [2] [p.186].

[5] $\{D_n\}_1^N$ is not necessary tiling, $\check{D}_n$ may overlap.

Lemma 4.2. *If the contractivity factor of the fractal transform operator Θ is $\lambda < 1$ and $\nu = 1/(1-\lambda)$, then Θ is mapping the image space $IM_D^{\alpha,\nu}$ in itself.*

Proof. Let $f, g \in IM_D^{\alpha,\nu}$. Then, according to the First Tiling Lemma and Lemma 4.1,

$$r(D,\alpha;\Psi(f),\Psi(g)) \leq \max_{1\leq n\leq N} r(D_n,\alpha;\Lambda(v_n,w_n)(f),\Lambda(v_n,w_n)(g))$$
$$\leq \lambda r(D,\alpha;f,g),$$

since Ψ is contractive operator with contractivity factor $\lambda < 1$.

According to Second Tiling Lemma and Lemma 4.1, for every $f \in IM_D^{\lambda,\nu}$,

$$\tau(D,\alpha,\Psi(f);\delta) \leq \alpha^{-1}\delta + \max_{1\leq n\leq N} \tau(D_n,\alpha,\Lambda(v_n,w_n)(f);\delta)5.32$$
$$\leq \alpha^{-1}\delta + \lambda\tau(D,\alpha,f;\delta),$$

and as $\lambda = 1 - 1/\nu$ and f is of the class ν, we get

$$\tau(D,\alpha,\Psi(f);\delta) \leq \alpha^{-1}\delta + \lambda\alpha^{-1}\nu\delta = \alpha^{-1}\nu\delta,$$

and the Lemma is proved for the operator Ψ.

In the same way we consider the operator Ξ.

For operator Φ, from Lemma 3.4, Lemma 4.1 and Corollary 4.1, we have

$$r(D,\alpha;\Phi(f),\Phi(g)) \leq \sum_{n=1}^{N} r(D_n,\alpha;p_n\Lambda(v_n,p_nw_n)(f),p_n\Lambda(v_n,p_nw_n)(g))$$
$$\leq \lambda r(D,\alpha;f,g)\sum_{n=1}^{N} p_n = \lambda r(D,\alpha;f,g),$$

since Φ is contractive operator with contractivity factor $\lambda < 1$.

According to Second Tiling Lemma, Lemma 4.1 and Corollary 4.1, for every $f \in IM_D^{\lambda,\nu}$,

$$\tau(D,\alpha,\Phi(f);\delta) \leq \alpha^{-1}\delta + \sum_{n=1}^{N} \tau(D_n,\alpha,p_n\Lambda(v_n,p_nw_n)(f);\delta)5.33$$
$$\leq \alpha^{-1}\delta + \lambda\tau(D,\alpha,f;\delta),$$

and as $\lambda = 1 - 1/\nu$ and f is of the class ν, we get

$$\tau(D,\alpha,\Phi(f);\delta) \leq \alpha^{-1}\nu\delta,$$

and the Lemma is proved also for the operator Φ.

In the same way we consider the operator Υ and complete the proof.

From Theorem 4.2 and Lemma 4.2 we get finally the following result.

Theorem 4.4. (Theorem for fractal transforms) *Let Θ be a fractal transform operator with a contractivity factor $\lambda < 1$, and let $\nu = 1/(1-\lambda)$. Then:*

i) For any function $f_0 \in IM_D^{\alpha,\nu}$, the iterative sequence

$$\{f_m\}_0^\infty; \quad f_m = \Theta(f_{m-1}), \; m = 1, 2, 3, \ldots$$

is convergent.

ii) The operator Θ has a unique eigen-function $f_\Theta \in IM_D^{\lambda,\nu}$ and $f_\Theta = \lim_{i\to\infty} f_m$.

iii) For every natural m we have

$$r(D, \lambda; f_\Theta, f_m) \le \frac{\lambda^m M}{1-\lambda} r(D, \alpha; f_0, f_1).$$

Corollary 4.2. *Let Θ is a fractal transform operator with contractivity factor $\lambda < 1$, then for any function $f_0 \in IM_D^\alpha$, the iterative sequence*

$$\{f_m\}_0^\infty; \quad f_m = \Theta(f_{m-1}), \; m = 1, 2, 3, \ldots$$

is convergent, and the limit f_Θ of this sequence is a function from $IM_D^{\alpha,\nu}$, where $\nu = 1/(1-\lambda)$.

P r o o f. Having in mind the proof of Lemma 4.2 and especially the inequalities 4.32 and 4.33, we have

$$\tau(D, \alpha, f_m; \delta) \le \alpha^{-1}\delta(1 + \lambda + \lambda^2 + \ldots + \lambda^{m-1}) + \lambda^m \tau(D, \alpha, f_0; \delta)$$

and consequently

$$\tau(D, \alpha, f_\Theta; \delta) \le \alpha^{-1}\delta/(1-\lambda),$$

which completes the proof.

5.3 Pixel functions

For simplicity, in this section we shall use the Hausdorff distance with parameter $\alpha = 1$ and the notations $r(\alpha) = r$, $IM_D^\alpha = IM_D$.

For every natural number s we define the partition

$$D(s) = \{d_{i,j}^s = \{(x, y) : x \in (i2^{-s}, (i+1)2^{-s}), \; y \in (j2^{-s}, (j+1)2^{-s})\}$$

$$: i, j = 0, 1, 2, \ldots, 2^s - 1\}.$$

The squares $d_{i,j}$ are called pixels.

Definition 4.8. A function $f \in IM_D$ is called pixel function with resolution s if f is constant in every open square $d^s_{i,j}$ of the partition $D(s)$.

The set of the pixel functions with resolution s in IM_D is denoted by $IM_{D(s)}$.

The set $IM_{D(s)}$ is a complete subset of the metric space IM_D. Obviously, $IM_{D(s)}$ is a 2^{2s} dimentional linear subspace of IM_D.

Let us mention, that a function $f \in IM_{D(s)}$ is completly defined if it is defined in the center (ξ_i, η_j) of every pixel $d_{i,j}$; $i, j = 0, 1, 2, \ldots, 2^s - 1$. Every function $f \in IM_{D(s)}$ is defined by 2^{2s} real numbers

$$f_{i,j} = f(\xi_i, \eta_j);\ i, j = 0, 1, 2, \ldots, 2^s - 1.$$

The metric subspaces $IM_{D(s)}$ are included in each other, or

$$IM_{D(s)} \subset IM_{D(s+1)};\ s = 1, 2, 3, \ldots.$$

Definition 4.9. For every function $f \in IM_D$ we define the so-called pixeling operator $P_s(f) \in IM_{D(s)}$ in the fopllowing way

$$P_s(f; x) = \begin{cases} \int_{d^s_{i,j}} f(x) \colon dx & \text{if} x \in \bigcup_{i,j=0}^{2^s-1}, \\ \text{as an element of } IM_D & \text{if} x \in \bigcup_{i,j=0}^{2^s-1} \partial D_{I,J}. \end{cases}$$

It is obvious that the pixeling operator is a projecting linear positive operator $P_s(f) : IM_D \to IN_{D(s)}$.

Let us mention, that the Hausdorff distance $r(f, P_s(f))$ between an image function f and its transform with the pixeling operator may be arbitrarily big. On the other hand, the physical transformation, digitizing the real images is a kind of integration. This fact calls for some thoughts for an integral variant of the Hausdorff metric.

Let $W \subset IM_D$ and

$$E(r, W; f) = \inf_{P \in W} r(f, P)$$

be the best approximation of the function f with elements from the set W with respect to the distance r.

Now we shall prove that the principle of *uniform digitizing* is valid for all functions from IM_D.

Theorem 4.5. *For every $f \in IM_D$*

$$E(r, IM_{D(s)}; f) \leq 2^{1-s}$$

and

$$d(IM_D, IM_{D(s)}) = \sup_{f \in IM_D} E(r, IM_{D(s)}; f) = 2^{1-s}.$$

Proof. Let $f \in IM_D$ and

$$\begin{aligned} m_{i,j} &= \inf\{f(x,y) : (x,y) \in \overline{d_{i,j}^{s-1}}\}, \\ M_{i,j} &= \sup\{f(x,y) : (x,y) \in \overline{d_{i,j}^{s-1}}\}; \\ i,j &= 0,1,2,\ldots,2^{s-1}-1, \end{aligned}$$

where $\overline{d_{i,j}^{s-1}}$ is the closure of $d_{i,j}^{s-1}$.

We define the function $h \in IM_{D(s)}$ in the following way. As we mentioned already, it is sufficient to determine the values $h_{i,j}$ of the function h in the centers of the pixels $d_{i,j}^s$. Then, let

$$\begin{aligned} h_{2i,2j} &= h_{2i+1,2j+1} = m_{i,j}, \\ h_{2i+1,2j} &= h_{2i,2j+1} = M_{i,j}; \\ i,j &= 0,1,2,\ldots,2^{s-1}-1. \end{aligned}$$

Let (x,y) be an arbitrary point from D and let $(x,y) \in \overline{d_{i,j}^{s-1}}$, hence $|x - i2^{-s+1} - 2^{-s}| \leq 2^{-s}$ and $|y - j2^{-s+1} - 2^{-s}| \leq 2^{-s}$. Then $m_{i,j} \leq f(x,y) \leq M_{i,j}$. From the definition of the function h it follows that the point $(i2^{-s+1} + 2^{-s}, j2^{-s+1} + 2^{-s}, f(x,y)) \in F(h)$. Therefore

$$\inf_{(\xi,\eta,\zeta) \in F(h)} \max[|f(x,y) - \zeta|, \max\{|x-\xi|, |y-\eta|\}] \leq 2^{-s}.$$

From the other hand, there exist two points $(\xi_1,\eta_1), (\xi_2,\eta_2) \in \overline{d_{i,j}^{s-1}}$ such that $(\xi_1,\eta_1,m_i), (\xi_2,\eta_2,M_i) \in F(f)$, therefore

$$\inf_{(\xi,\eta,\zeta) \in F(f)} \max[|h(x,y) - \zeta|, \max\{|x-\xi|, |y-\eta|\}] \leq 2^{-s+1}.$$

Consequently,

$$|f(x,y) \ominus h(x,y)| \leq 2^{-s+1} \text{ and } r(f,h) \leq 2^{-s+1}.$$

That completes the proof of the first part of the theorem.

To prove the second part, let us consider, for $\epsilon \in (0, 2^{-s-1})$, the function $g \in IM_D$ defined as follows

$$g_\epsilon(x,y) = \begin{cases} -1 & \text{for } (x,y) \in d'_\epsilon, \\ 1 & \text{for } (x,y) \in d''_\epsilon, \\ 0 & \text{for } (x,y) \in D \setminus (\overline{d'_\epsilon} \cup \overline{d''_\epsilon}), \end{cases}$$

where

$$d'_\epsilon = \{(x,y):\ x,y \in (0,\epsilon)\},\quad d''_\epsilon = \{(x,y):\ x,y \in (\epsilon, 2\epsilon)\}.$$

It is easy to see, that for every function $f \in IM_{D(s)}$, the inequality

$$r(g_\epsilon, f) \geq 2^{-s+1} - 2\epsilon$$

holds, and consequently

$$\overline{\lim_{\epsilon\to\infty}}\, r(g_\epsilon, f) = 2^{-s+1}.$$

From the last there follows the second statement of the theorem, which completes the proof.

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