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Optimal Solutions of Linear Tridiagonal Systems of Equations Involving Inexact Right-Hand Side

*N. S. Dimitrova**

Presented by Bl. Sendov

Considered is a method for determination of the optimal solution of a tridiagonal linear system of equations with uncertain (interval) right-hand side derived from the known sweep method by means of extended interval-arithmetic operations. An algorithm with result verification in computer interval arithmetic is formulated.

1. Introduction

We consider a system of linear equations of the form $Ax = b$ with a coefficient $n \times n$–matrix $A = (a_{ij})$ and an n–dimensional right-hand side vector $b = (b_i)$. Suppose that inclusion intervals A_{ij} for the components a_{ij} resp. B_i for b_i are known, that is $a_{ij} \in A_{ij}$, $b_i \in B_i$, $i, j = 1, 2, ..., n$. Denote by $[A] = (A_{ij})$ resp. $[b] = (B_i)$ the corresponding interval matrix resp. interval vector. Let $\{x\} = \{x : Ax = b, A \in [A], b \in [b]\}$ be the solution set of the linear system $Ax = b$ for $A \in [A]$, $b \in [b]$. It is known that $\{x\}$ can have a very complicated structure (see e.g. [11]). In the situation when $\{x\}$ is a bounded set (e.g. when every matrix $A \in [A]$ is nonsingular) the hull $[x]$ of $\{x\}$ [11] is defined as

$$[x] = [\inf\{x\}, \sup\{x\}].$$

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We shall call $[x]$ optimal solution of the interval linear system $[A]x = [b]$. Every interval vector $[x']$ with $[x'] \supseteq \{x\}$ we shall call a solution of the interval system $[A]x = [b]$.

The computation of x is a difficult problem. Some classes of matrices are known (e.g. the class of M-matrices, of totaly nonnegative matrices a.s.o., see [2], [6], [12]) for which the optimal solution can be computed using different methods.

A. Neumaier [10], [11] proposes a new approach for solving this problem. The computation of a solution of $[A]x = [b]$ is reduced to computations of solutions of two linear systems with uncertaintess in the right-hand side only in the following manner.

Let w be a positive real vector. Denote $[w] = [-1,1]w$. An interval norm $\|\cdot\|_w$ (see [10]) is defined by

$$\|[x]\|_w := \min\{a \geq 0 \colon x \subseteq a[w]\}, \ a \text{ real}.$$

Theorem 1. [10]. *Let be $[A] = [A] - [E]$ with a point matrix A and an interval matrix $[E]$. Denote by $[z]$ a solution of the interval system $A[z] = [E][w]$, i.e. $[z] \supseteq \{z \colon Az = b,\ b \in [E][w]\}$, so that $\|[z]\|_w = \alpha < 1$ holds. Denote further by $[y]$ a solution of $A[y] = [b]$, i.e. $[y] \supseteq \{y \colon Ay = b,\ b \in [b]\}$ and let be $\|[y]\|_w = \beta$. Then the following presentation for the solution $[x]$ of the interval system $[A]x = [b]$ holds:*

$$[x] = [y] + (\beta/(1-\alpha))[z].$$

From this theorem it follows that the computation of a solution of linear interval systems with interval right-hand side vector is very important. The present paper considers the computation of the optimal solution of a linear interval system in the special case of tridiagonal point coefficient matrix A using extended interval arithmetic.

Let R be the set of reals and IR the set of all compact intervals over R. An interval $X \in IR$ is denoted by $X = [\underline{x}, \bar{x}]$, $\underline{x} \leq \bar{x}$. The width of X is defined by $\omega(X) = \bar{x} - \underline{x}$. An interval with end-points α, $\beta \in R$ (where $\alpha \leq \beta$ is not necessarily) will be denoted by $[\alpha \vee \beta]$, that is $[\alpha \vee \beta] = \{[\alpha, \beta]$, if $\alpha \leq \beta$; $[\beta, \alpha]$, if $\alpha \geq \beta\}$.

For $X = [\underline{x}, \bar{x}]$, $Y = [\underline{y}, \bar{y}] \in IR$ we define the following extended interval-arithmetic operations which we need further ([3]-[5]):

$$
\begin{aligned}
X+Y &= [\underline{x}+\underline{y}, \bar{x}+\bar{y}];\\
X+^{-} Y &= [(\underline{x}+\bar{y}) \vee (\bar{x}+\underline{y})];\\
X-Y &= [\underline{x}-\bar{y}, \bar{x}-\underline{y}];\\
X-^{-} Y &= [(\underline{x}-\underline{y}) \vee \bar{x}-\bar{y})];\\
X \times Y &= [\min\{\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\}];\\
1/Y &= [1/\bar{y}, 1/\underline{y}], 0 \notin Y;\\
X/Y &= X \times (1/Y), 0 \notin Y.
\end{aligned}
$$

To compute the optimal solution of a tridiagonal linear system with interval right-hand side vector we shall use the theory for interval-arithmetic presentations of ranges of monotone functions of many variables [4] which we shortly introduce below.

Denote by Θ the set $\Theta = \{(\theta_1, \theta_2, ..., \theta_n): \ \theta_i \in \{\leq, \geq\}\}$, where $\leq$ and $\geq$ are the usual order relations in R^n. The set Θ consists of 2^n elements which will be called order relations in R^n. We say that the vectors $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ are ordered and write $x\theta y$ if there is an order relation $\theta = (\theta_1, \theta_2, ..., \theta_n) \in \Theta$ such that $x_i \ \theta_i \ y_i$ for $i = 1, 2, ..., n$.

A function $f\colon D \to R$, $D \subset R^n$, is called monotone in D, if there is an order relation $\theta = (\theta_1, \theta_2, ..., \theta_n) \in \Theta$ such that for every $x, y \in D$ with $x\theta y$ it follows $f(x) \leq f(y)$. The set of all monotone functions on D will be denoted by $\mathcal{M} = \mathcal{M}(D)$.

Let $f \in \mathcal{M}$ and $[x] = [\underline{x}, \bar{x}] = ([\underline{x}_i, \bar{x}_i])_{i=1}^n$ be an interval vector containd in D, i.e. $[x] \subset D$. The order relation $\theta = (\theta_1, \theta_2, ..., \theta_n)$ indicates two real vectors $x^- = (x_1^-, x_2^-, ..., x_n^-)$ and $x^+ = (x_1^+, x_2^+, ..., x_n^+)$ with

$$
x_i^- = \begin{cases} \underline{x}_i & \text{if } \theta_i = \leq, \\ \bar{x}_i & \text{if } \theta_i = \geq; \end{cases} \qquad x_i^+ = \begin{cases} \bar{x}_i & \text{if } \theta_i = \leq, \\ \underline{x}_i & \text{if } \theta_i = \geq . \end{cases}
$$

Then the range $f([x]) = [\inf_{x\in[x]} f(x), \sup_{x\in x} f(x)]$ of f on $[x]$ is obviously presented by the interval $[f(x^-), f(x^+)]$.

Assume that the functions f, f_1, $f_2 \in \mathcal{M}$ and $f = f_1 * f_2$, $* \in \{+, -\}$. We are interested in presenting the range $f([x])$ by means of f_1 and f_2.

Denote $\tilde{x} = \tilde{x}(f; [x]) = co\{x^-, x^+\} = \{tx^- + (1-t)x^+ : t \in [0, 1]\}$ and consider the intervals

$$
f_i(\tilde{x}) = [f_i(x^-) \vee f_i(x^+)], \quad i = 1, 2;
$$

define the values

$$\delta(f_i(\tilde{x})) = f_i(x^-) - f_i(x^+), \quad i = 1, 2.$$

Obviously the following equalities hold true:

$$|\delta(f_i(\tilde{x}))| = \omega(f_i(\tilde{x}));$$

$$\delta(\alpha f_i(\tilde{x}) \pm \beta f_i(\tilde{x})) = \alpha\delta(f_i(\tilde{x})) \pm \beta\delta(f_i(\tilde{x})), \quad \alpha, \beta \in R.$$

Theorem 2. [4]. *Let f, f_1, $f_2 \in \mathcal{M}$ and $f = f_1 * f_2$, $* \in \{+, -\}$. Then the following interval - arithmetic presentation for the range of f on $[x]$ holds true:*

i) $f(x) = \begin{cases} f_1(\tilde{x}) + f_2(\tilde{x}) \text{ if } \delta(f_1(\tilde{x}))\delta(f_2(\tilde{x})) \geq 0, \\ f_1(\tilde{x}) +^- f_2(\tilde{x}) \text{ if } \delta(f_1(\tilde{x}))\delta(f_2(\tilde{x})) < 0; \end{cases}$

ii) $f(x) = \begin{cases} f_1(\tilde{x}) -^- f_2(\tilde{x}) \text{ if } \delta(f_1(\tilde{x}))\delta(f_2(\tilde{x})) \geq 0, \\ f_1(\tilde{x}) - f_2(\tilde{x}) \text{ if } \delta(f_1(\tilde{x}))\delta(f_2(\tilde{x})) < 0; \end{cases}$

2. Computation of the optimal solution of a tridiagonal linear system with uncertain right-hand side.

We consider the linear system

$$(1) \qquad Ay = d$$

with a regular coefficient matrix A of the form

$$\begin{pmatrix} 1 & a_0 & 0 & 0 & \dots & 0 & 0 \\ b_1 & 1 & a_1 & 0 & \dots & 0 & 0 \\ 0 & b_2 & 1 & a_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & b_{n-1} & 1 & a_{n-1} \\ 0 & \dots & \dots & 0 & 0 & b_n & 1 \end{pmatrix}$$

and a right-hand side vector $d = (d_0, d_1, ..., d_n)^T$. Assume that $d \in [d] = (D_i)_{i=0}^n = ([\underline{d}_i, \bar{d}_i])_{i=0}^n$ and consider the interval tridiagonal system

$$(2) \qquad A[y] = [d].$$

We shall compute the optimal solution $[y] = [\inf\{y\}, \sup\{y\}]$ of (2). It should be mentioned that the solution set $\{y\}$ of (2) is convex and bounded.

From the numerical algebra the sweep method for solving (1) is known: the solution $y = (y_0, y_1, ..., y_n)$ is seaked in the form

$$y_i = p_i y_{i+1} + q_i, \quad i = 0, 1, ..., n-1, \tag{3}$$

where the coefficients p_i and q_i are determined by the following recursion formulae:

$$p_0 = -a_0,$$

$$p_i = -a_i/(1 + p_{i-1} b_i), \quad i = 1, 2, ..., n; \tag{4}$$

$$q_0 = d_0,$$

$$q_i = (d_i - b_i q_{i-1})/(1 + p_{i-1} b_i), \quad i = 1, 2, ..., n-1. \tag{5}$$

Using (3) for $i = n - 1$ and the last equation of (1), i.e. from the system

$$\begin{cases} b_n y_{n-1} + y_n = d_n \\ y_{n-1} - p_{n-1} y_n = q_{n-1} \end{cases} \tag{6}$$

we compute y_n using e.g. the Cramer's formulae. The components y_{n-1}, y_{n-2}, ..., y_0 are then determined consecutively using (3).

The applying of this method in the interval computation leads to a method for determination of the optimal solution $[y] = (Y_0, Y_1, ..., Y_0)$ of (2). The idea is based on the computation of the ranges of the functions $q_i = q_i(d_0, d_1, ...d_i)$, $i = 0, 1, ..., n-1$, and $y_j = y_j(d_0, d_1, ..., d_n)$, $j = n, n-1, ..., 0$, considered as functions of the variables d_0 , d_1,...,d_n over the interval vector $[d]$. The coefficients p_i do not depend on d_j, i.e. they are real numbers.

We next formulate two propositions.

Proposition 1. [1; Chpt.17]. *Let $A^{-1} = (a_{ij}^{(-1)})_{i,j=0}^{n}$ be the inverse matrix of A. Then the following presentation holds true:*

$$\frac{\partial y_i}{\partial d_j} = a_{ij}^{(-1)}, \quad i, j = 0, 1, ..., n.$$

Proposition 2. *Let the coefficients p_i be determened by (4). Denote*

$$s_0 = -b_1,$$

$$s_i = -b_{i+1}/(1 + p_{i-1}b_i), \quad i = 1, 2, ..., n-1.$$

Then the elements $a_{ij}^{(-1)}$ *of the inverse matrix* $A^{(-1)}$ *are expressed by the following recursion formulae:*

$$a_{nn}^{(-1)} = 1/(1 + p_{n-1}b_n);$$
$$a_{ii}^{(-1)} = p_i s_i a_{i+1i+1}^{(-1)} + 1/(1 + p_{i-1}b_i), \quad i = n-1, n-2, ..., 1, 0;$$

$$a_{ik}^{(-1)} = \begin{cases} p_i p_{i+1} ... p_{k-1} a_{kk}^{(-1)} & \text{if } i < k, \\ s_k s_{k+1} ... s_{i-1} a_{ii}^{(-1)} & \text{if } i > k, \end{cases}$$

where $p_{-1} = b_0 = 0$.

Similar formulae for $a_{ij}^{(-1)}$ can be found in [13].

Denote by Q_i resp. Y_j the ranges of the functions $q_i = q_i(d_o, d_1, ..., d_i)$ resp. $y_j = y_j(d_0, d_1, ..., d_n)$ on $[d]$, $i = 0, 1, ..., n-1$, $j = 0, 1, ..., n$. Obviously, the following interval-arithmetic presentation of Q_i is valid (see [9]):

$$Q_i = (D_i - b_i Q_{i-1})/(1 + p_{i-1}b_i), i = 0, 1, ..., n.$$

Using (6) we obtain for y_n

$$y_n = (d_n - b_n q_{n-1})/(1 + p_{n-1}b_n)$$

and the range Y_n can be expressed by

$$Y_n = (D_n - b_n Q_{n-1})/(1 + p_{n-1}b_n).$$

For the interval-arithmetic presentation of the range Y_k, $k = n-1, n-2, ..., 0$, we shall use Theorem 2, *i)* with $f_1 = p_k y_{k+1}$, $f_2 = q_k$ and $f = f_1 + f_2 = y_k$. For this purpose we need firstly the real vectors

$$d^-(y_k) = (d_0^-(y_k), d_1^-(y_k), ..., d_n^-(y_k)),$$
$$d^+(y_k) = (d_0^+(y_k), d_1^+(y_k), ..., d_n^+(y_k)).$$

Using Proposition 1 we obtain

$$d_i^-(y_k) = \begin{cases} \underline{d}_i & \text{if } a_{ki}^{(-1)} \geq 0, \\ \bar{d}_i & \text{otherwise;} \end{cases} \qquad d_i^+(y_k) = \begin{cases} \bar{d}_i & \text{if } a_{ki}^{(-1)} \geq 0, \\ \underline{d}_i & \text{otherwise;} \end{cases}$$

$$i = 0, 1, ..., n.$$

Denote further

$$\tilde{d}_k = co\{d^-(y_k), d^+(y_k)\} = \{td^-(y_k) + (1-t)d^+(y_k) \colon t \in [0,1]\}.$$

Using Theorem 2, *i)* we thus obtain

$$Y_k = \begin{cases} p_k Y_{k+1}(\tilde{d}_k) + Q_k(\tilde{d}_k) & \text{if } p_k\delta(Y_{k+1}(\tilde{d}_k))\delta(Q_k(\tilde{d}_k)) \geq 0, \\ p_k Y_{k+1}(\tilde{d}_k) +^- Q_k(\tilde{d}_k) & \text{otherwise,} \end{cases}$$

where $Y_{k+1}(\tilde{d}_k)$ resp. $Q_k(\tilde{d}_k)$ mean the intervals

$$Y_{k+1}(\tilde{d}_k) = [y_{k+1}(d^-(y_k)) \vee y_{k+1}(d^+(y_k))],$$
$$Q_k(\tilde{d}_k) = [q_k(d^-(y_k)) \vee q_k(d^+(y_k))].$$

The above expression for Y_k contains the intervals $Q_k(\tilde{d}_k)$ and $Y_{k+1}(\tilde{d}_k)$. Since the functions q_i are computed recursively according (5) we need for the presentation of $Q_k(\tilde{d}_k)$ the intervals $Q_0(\tilde{d}_k)$, $Q_1(\tilde{d}_k)$, ..., $Q_{k-1}(\tilde{d}_k)$. To compute $Y_{k+1}(\tilde{d}_k)$ we need $Q_{k+1}(\tilde{d}_k)$, ... , $Q_{n-1}(\tilde{d}_k)$ and also $Y_n(\tilde{d}_k)$, $Y_{n-1}(\tilde{d}_k)$, ... ,$Y_{k+2}(\tilde{d}_k)$.

Proposition 3. *For $j = 0, 1, ..., n$, and for some $k, 0 \leq k \leq n-1$, $D_j(\tilde{d}_k) = D_j$ and $\delta(D_j) = -\mathrm{sgn}(a_{kj}^{-1})w(D_j)$ holds true.*

Proof. We have

$$D_j(\tilde{d}_k) = [d_j(d_k^-(y_k)) \vee d_j(d_k^+(y_k))]$$

$$= \begin{cases} \{d_j : d_j = (\underline{d}_j - \bar{d}_j)t + \bar{d}_j, \quad t \in [0,1]\} \text{ if } a_{kj}^{(-1)} \geq 0, \\ \{d_j : d_j = (\bar{d}_j - \underline{d}_j)t + \underline{d}_j, \quad t \in [0,1]\} \text{ otherwise;} \end{cases}$$

$$= \begin{cases} [\{d_j = (\underline{d}_j - \bar{d}_j)t + \bar{d}_j, t = 1\}, \{d_j = (\underline{d}_j - \bar{d}_j)t + \bar{d}_j, t = 0\}] & \text{if } a_{kj}^{(-1)} \geq 0, \\ [\{d_j = (\bar{d}_j - \underline{d}_j)t + \underline{d}_j, t = 0\}, \{d_j = (\bar{d}_j - \underline{d}_j)t + \underline{d}_j, t = 1\}] & \text{otherwise;} \end{cases}$$

$$= \begin{cases} [\underline{d}_j, \bar{d}_j] & \text{if } a_{kj}^{(-1)} \geq 0, \\ [\underline{d}_j, \bar{d}_j] & \text{otherwise;} \end{cases}$$

$$= D_j.$$

According to the above presentation we obtain

$$\delta(D_j) = d_j(d_k^-(y_k)) - d_j(d_k^+(y_k))$$

$$= \begin{cases} \{d_j = (\underline{d}_j - \bar{d}_j)t + \bar{d}_j, t = 1\} - \{d_j = (\underline{d}_j - \bar{d}_j)t + \bar{d}_j, t = 0\} & \text{if } a_{kj}^{(-1)} \geq 0, \\ [\{d_j = (\bar{d}_j - \underline{d}_j)t + \underline{d}_j, t = 1\} - \{d_j = (\bar{d}_j - \underline{d}_j)t + \underline{d}_j, t = 0\}] & \text{otherwise;} \end{cases}$$

$$= \begin{cases} -\omega(D_j) & \text{if } a_{kj}^{(-1)} \geq 0, \\ \omega(D_j) & \text{otherwise;} \end{cases}$$

$$= -\,\mathrm{sgn}(a_{kj}^{(-1)})\omega(D_j).$$

Proposition 4. *For some k, $0 \leq k \leq n-1$, and for $i = 0, 1, 2, ..., k$, the presentation $Q_i(\tilde{d}_k) = Q_i = (D_i - b_i Q_{i-1})/(1 + p_{i-1} b_i)$ is valid.*

Proof. We have for $i = 0$:

$$Q_0(\tilde{d}_k) = D_0(\tilde{d}_k) = D_0 = Q_0.$$

For $i = 1$ we obtain

$$\begin{aligned} Q_1(\tilde{d}_k) &= [q_1(d^-(y_k)) \vee q_1(d^+(y_k))] \\ &= \begin{cases} (1/(1+p_0 b_1))[\underline{d}_1 - b_1 \underline{d}_0, \bar{d}_1 - b_1 \bar{d}_0] & \text{if } a_{k0}^{(-1)} \geq 0, a_{k1}^{(-1)} \geq 0; \\ (1/(1+p_0 b_1))[\bar{d}_1 - b_1 \underline{d}_0, \underline{d}_1 - b_1 \bar{d}_0] & \text{if } a_{k0}^{(-1)} \geq 0, a_{k1}^{(-1)} < 0; \\ (1/(1+p_0 b_1))[\underline{d}_1 - b_1 \bar{d}_0, \bar{d}_1 - b_1 \underline{d}_0] & \text{if } a_{k0}^{(-1)} < 0, a_{k1}^{(-1)} \geq 0; \\ (1/(1+p_0 b_1))[\bar{d}_1 - b_1 \bar{d}_0, \underline{d}_1 - b_1 \underline{d}_0] & \text{if } a_{k0}^{(-1)} < 0, a_{k1}^{(-1)} < 0; \end{cases} \\ &= \begin{cases} (1/(1+p_0 b_1))(D_1 -^- b_1 D_0) & \text{if } b_1 a_{k1}^{(-1)} a_{k0}^{(-1)} \geq 0, \\ (1/(1+p_0 b_1))(D_1 - b_1 D_0) & \text{if } b_1 a_{k1}^{(-1)} a_{k0}^{(-1)} < 0. \end{cases} \end{aligned}$$

Using Proposition 2 we thus obtain

$$\begin{aligned} b_1 a_{k1}^{(-1)} a_{k0}^{(-1)} &= b_1 s_1 ... s_{k-1} a_{kk}^{(-1)} s_0 s_1 ... s_{k-1} a_{kk}^{(-1)} \\ &= -(s_0 s_1 ... s_{k-1} a_{kk}^{(-1)})^2 < 0, \end{aligned}$$

from what it follows

$$Q_1(\tilde{d}_k) = (1/(1+p_0 b_1))(D_1 - b_1 Q_0) = Q_1.$$

Further, using Proposition 3,

$$\begin{aligned} \delta(Q_1) &= (1/(1+p_0 b_1))(\delta(D_1) - b_1 \delta(Q_0)) \\ &= (1/(1+p_0 b_1))(-\mathrm{sgn}(a_{k1}^{(-1)})\omega(D_1) + b_1 \mathrm{sgn}(a_{k0}^{(-1)})\omega(D_0)) \\ &= -\mathrm{sgn}(a_{k1}^{(-1)}/(1+p_0 b_1))(\omega(D_1) - b_1 \mathrm{sgn}(a_{k0}^{(-1)})\mathrm{sgn}(a_{k1}^{(-1)})\omega(D_0)), \end{aligned}$$

hence $\mathrm{sgn}\,(\delta(Q_1)) = -\mathrm{sgn}(a_{k1}^{(-1)}/(1+p_0 b_1))$.

Suppose that for some i, $1 < i < k$,

$$\begin{aligned} &Q_i(\tilde{d}_k) = Q_i = (1/(1+p_{i-1} b_i))(D_i - b_i Q_{i-1}), \\ &\mathrm{sgn}(\delta(Q_i)) = -\mathrm{sgn}(a_{ki}^{(-1)}/(1+p_{i-1} b_i)) \end{aligned}$$

hold. We shall prove that

$$Q_{i+1}(\tilde{d}_k) = Q_{i+1} = (1/(1+p_i b_{i+1}))(D_{i+1} - b_{i+1}Q_i),$$
$$\mathrm{sgn}(\delta(Q_{i+1})) = -\mathrm{sgn}(a_{ki+1}^{(-1)}/(1+p_i b_{i+1})).$$

Indeed,

$$Q_{i+1}(\tilde{d}_k) = \begin{cases} (1/(1+p_i b_{i+1}))(D_i -^- b_{i+1}Q_i) & \text{if } b_{i+1}\delta(D_{i+1})\delta(Q_i) \geq 0, \\ (1/(1+p_i b_{i+1}))(D_i - b_{i+1}Q_i) & \text{if } b_{i+1}\delta(D_{i+1})\delta(Q_i) < 0. \end{cases}$$

Using Proposition 2 and 3 we obtain

$$\mathrm{sgn}(b_{i+1}\delta(D_{i+1})\delta(Q_i)) = \mathrm{sgn}(b_{i+1}a_{ki+1}^{(-1)}\omega(D_{i+1})a_{ki}^{(-1)}/(1+p_{i-1}b_i))$$
$$= \mathrm{sgn}(-s_i a_{ki+1}^{(-1)}a_{ki}^{(-1)}) = -\mathrm{sgn}(a_{ki+1}^{(-1)})^2 < 0$$

and therefore

$$Q_{i+1}(\tilde{d}_k) = Q_{i+1}.$$

Further we have

$$\delta(Q_{i+1}) = (1/(1+p_i b_{i+1}))(\delta(D_{i+1}) - b_{i+1}\delta(Q_i))$$
$$= -\mathrm{sgn}(a_{ki+1}^{(-1)}/(1+p_i b_{i+1}))(\omega(D_{i+1}) = \mathrm{sgn}(a_{ki+1}^{(-1)})b_{i+1}\ zd(Q_i)),$$

$$\mathrm{sgn}(a_{ki+1}^{(-1)}b_{i+1}\delta(Q_i)) = \mathrm{sgn}(a_{ki+1}^{(-1)}).\mathrm{sgn}(b_{i+1}).\mathrm{sgn}(\delta(Q_i))$$
$$= \mathrm{sgn}(a_{ki+1}^{(-1)}).\mathrm{sgn}(b_{i+1}).\mathrm{sgn}(a_{ki}^{(-1)}/(1+p_{i-1}b_i))$$
$$= \mathrm{sgn}(a_{ki+1}^{(-1)}).\mathrm{sgn}(s_i).$$
$$sgn(a_{ki}^{(-1)}))$$
$$= \mathrm{sgn}(a_{ki+1}^{(-1)})^2 \geq 0,$$

hence

$$\mathrm{sgn}(\delta(Q_{i+1})) = -\mathrm{sgn}(a_{ki+1}^{(-1)}/(1+p_i b_{i+1})),$$

which proves the Proposition. ■

Corollary. *Under the assumptions of Proposition 4, the equality*

$$\mathrm{sgn}(\delta(Q_k)) = -\mathrm{sgn}(a_{kk}^{(-1)}/(1+p_{k-1}b_k))$$

holds true.

The algorithm for computing the optimal solution of the tridiagonal interval system (2), which can be considered as a sweep interval method for solving (2), is presented by the following

Theorem 3. *The optimal solution $[y] = (Y_0, Y_1, ..., Y_n)$ of the tridiagonal interval system (2) can be computed by the following algorithm $(A1) - (A2)$:*

(A1) For $j = 0$ to $n-1$ compute p_j according (4); end;
$Q_0 = D_0$;
For $j = 1$ to $n-1$ compute
$Q_j = (1/(1+p_{j-1}b_j))(D_j - b_j Q_{j-1})$;
$\mathrm{sgn}(\delta(Q_j)) = -\mathrm{sgn}(a_{jj}^{(-1)}/(1+p_{j-1}b_j))$;
end;
(A2) Compute $Y_n = (1/(1+p_{n-1}b_n))(D_n - b_n Q_{n-1})$;
For $k = n-1$ downto 0 do begin
For $j = 0$ to n compute
$\delta(D_j) = -\mathrm{sgn}(a_{kj}^{(-1)})\omega(D_j)$;
end;
For $j = k+1$ to $n-1$ compute

$$Q_j(\tilde{d}_k) = \begin{cases} (1/(1+p_{j-1}b_j))(D_j -^- b_j Q_{j-1}(\tilde{d}_k)) & \text{if } b_j\delta(D_j)\delta(Q_{j-1}(\tilde{d}_k)) \geq 0, \\ (1/(1+p_{j-1}b_j))(D_j - b_j Q_{j-1}(\tilde{d}_k)) & \text{otherwise;} \end{cases}$$

$\delta(Q_j(\tilde{d}_k)) = (1/(1+p_{j-1}b_j))(\delta(D_j) - b_j\delta(Q_{j-1}(\tilde{d}_k)))$;
$\{Q_k(\tilde{d}_k) = Q_k\}$;
end;
Compute

$$Y_n(\tilde{d}_k) = \begin{cases} (1/(1+p_{n-1}b_j))(D_n -^- b_n Q_{n-1}(\tilde{d}_k)) & \text{if } b_n\delta(D_n)\delta(Q_{n-1}(\tilde{d}_k)) \geq 0, \\ (1/(1+p_{n-1}b_n))(D_n - b_n Q_{n-1}(\tilde{d}_k)) & \text{otherwise;} \end{cases}$$

$\delta(Y_n(\tilde{d}_k)) = (1/(1+p_{n-1}b_n))(\delta(D_n) - b_n\delta(Q_{n-1}(\tilde{d}_k)))$;
For $j = n-1$ downto $k+1$ compute

$$Y_j(\tilde{d}_k) = \begin{cases} p_j Y_{j+1}(\tilde{d}_k) + Q_j(\tilde{d}_k) & \text{if } p_j\delta(Y_{j+1}(\tilde{d}_k))\delta(Q_j(\tilde{d}_k)) \geq 0, \\ p_j Y_{j+1}(\tilde{d}_k) +^- Q_j(\tilde{d}_k) & \text{otherwise;} \end{cases}$$

$\delta(Y_j(\tilde{d}_k)) = p_j\delta(Y_{j+1}(\tilde{d}_k)) + \delta(Q_j(\tilde{d}_k))$;
end;

$$Y_k = \begin{cases} p_k Y_{k+1}(\tilde{d}_k) + Q_k & \text{if } a_k a_{kk}^{(-1)}\delta(Y_{k+1}(\tilde{d}_k)) \geq 0, \\ p_k Y_{k+1}(\tilde{d}_k) +^- Q_k & \text{otherwise;} \end{cases}$$

end.

Proof. The optimality of $[y] = (Y_0, Y_1, ..., Y_n)$ follows from Theorem 2 and Moore's theorem [9]. The right-hand side condition in the computational formula for Y_k follows from the relation

$$\begin{aligned} \mathrm{sgn}(p_k\delta(Y_{k+1}(\tilde{d}_k))\delta(Q_k)) &= \mathrm{sgn}(p_k a_{kk}^{(-1)}/(1+p_{k-1}b_k)\delta(Y_{k+1}(\tilde{d}_k))) \\ &= -\mathrm{sgn}(a_k/(1+p_{k-1}b_k)^2 a_{kk}^{(-1)}\delta(Y_{k+1}(\tilde{d}_k))) \\ &= \mathrm{sgn}(a_k a_{kk}^{(-1)}\delta(Y_{k+1}(\tilde{d}_k))). \end{aligned}$$

Remark. In the algorithm (A1)—(A2) we need only the signs of the elements $a_{ij}^{(-1)}$ and not their values. For this purpose we can use any interval method (see e.g. [1], Chpt. 18) to compute enclosing intervals $A_{ij}^{(-1)} \ni a_{ij}$. In the situation when $0 \notin A_{ij}^{(-1)}$ we can determine the signs of the elements a_{ij}.

As a special case of Theorem 3 we obtain the following

Theorem 4. *Let the components of the matrix A and $a_{ij}^{(-1)}$ of the inverse matrix A^{-1} satisfy the conditions*

$$a_k b_{k+1} a_{kk}^{(-1)} a_{k+1k+1}^{(-1)} \geq 0,$$

$$a_{kk}^{(-1)}/(1 + p_{k+1} b_k) \geq 0, \quad k = 0, 1, ..., n-1$$

with $p_{-1} = b_0 = 0$. Then the optimal solution $[y] = (Y_0, Y_1, ..., Y_n)$ is computed by the following algorithm (B1)—(B2):

(B1) $p_0 = -a_0, \quad Q_0 = D_0$;
For $i = 1$ to $n-1$ compute
$p_i = -a_i/(1 + p_{i-1} b_i)$,
$Q_i = (1/(1 + p_{i-1} b_i))(D_i - b_i Q_{i-1})$;
end;

(B2) $Y_n = (1/(1 + p_{n-1} b_n))(D_n - b_n Q_{n-1})$;
For $k = n-1$ downto 0 compute
$Y_k = p_k Y_{k+1} + Q_k$;
end.

Proof. We shall prove the following relations:

1) $Y_{n-1} = p_{n-1} Y_n + Q_{n-1}$ if $a_{n-1} b_n a_{n-1n-1}^{(-1)} a_{nn}^{(-1)} \geq 0$,
$a_{n-1n-1}^{(-1)}/(1 + p_{n-2} b_{n-1}) \geq 0$;

2) $Y_{n-2} = p_{n-2} Y_{n-1} + Q_{n-1}$ if $a_{n-1} b_n a_{n-1n-1}^{(-1)} a_{nn}^{(-1)} \geq 0$,
$a_{n-2} b_{n-1} a_{n-2n-2}^{(-1)} a_{n-1n-1}^{(-1)} \geq 0$,
$a_{n-2n-2}^{(-1)}/(1 + p_{n-3} b_{n-2}) \geq 0$,
$a_{n-1n-1}^{(-1)}/(1 + p_{n-2} b_{n-1}) \geq 0$;

⋮

n-k-1) $Y_{k+1} = p_{k+1} Y_{k+2} + Q_{k+1}$ if $a_{k+1} b_{k+2} a_{k+1k+1}^{(-1)} a_{k+2k+2}^{(-1)} \geq 0$,
$a_{k+2} b_{k+3} a_{k+2k+2}^{(-1)} a_{k+3k+3}^{(-1)} \geq 0$,
...
$a_{n-1} b_n a_{n-1n-1}^{(-1)} a_{nn}^{(-1)} \geq 0$,
$a_{k+1k+1}/(1 + p_k b_{k+1}) \geq 0$,
...
$a_{n-1n-1}^{(-1)}/(1 + p_{n-2} b_{n-1}) \geq 0$;

n-k) $Y_k = p_k Y_{k+1} + Q_k$ if $a_k b_{k+1} a_{kk}^{(-1)} a_{k+1k+1}^{(-1)} \geq 0,$

$$a_{k+1} b_{k+2} a_{k+1k+1}^{(-1)} a_{k+2k+2}^{(-1)} \geq 0,$$
$$\dots$$
$$a_{n-1} b_n a_{n-1n-1}^{(-1)} a_{nn}^{(-1)} \geq 0,$$
$$a_{kk}/(1 + p_{k-1} b_k) \geq 0,$$
$$a_{k+1k+1}^{(-1)}/(1 + p_k b_{k+1}) > 0,$$
$$\dots$$
$$a_{n-1n-1}^{(-1)}/(1 + p_{n-2} b_{n-1}) \geq 0;$$

$k = n-4, n-3, ..., 1, 0.$

We see that the right-hand side conditions on step $(n-k-1)$ are included in the conditions on $(n-k)$-th step, exclusively the two inequalities

$$a_k b_{k+1} a_{kk}^{(-1)} a_{k+1k+1}^{(-1)} \geq 0 \text{ and}$$
$$a_{kk}^{(-1)}/(1 + p_{k-1} b_k) \geq 0,$$

which verification proves the theorem.

Without loss of generality we shall prove 1) and 2). Applying Theorem 3 with $n = k - 1$ to 1) we obtain

$$Y_n(\tilde{d}_{n-1}) = (1/(1 + p_{n-1} b_n))(D_n - b_n Q_{n-1}) \text{ if } b_n a_{n-1n}^{(-1)} \delta(Q_{n-1}) \geq 0.$$

Because of $\mathrm{sgn}(\delta(Q_{n-1})) = -\mathrm{sgn}(a_{n-1n-1}^{(-1)}/(1 + p_{n-2} b_{n-1}))$ we have according Proposition 2 and (5)

$$\mathrm{sgn}(b_n a_{n-1n}^{(-1)} \delta(Q_{n-1})) = -\mathrm{sgn}(b_n p_{n-1} a_{nn}^{(-1)} a_{n-1n-1}^{(-1)}/(1 + p_{n-2} b_{n-1}))$$
$$= \mathrm{sgn}(b_n (a_{n-1}/(1 + p_{n-2} b_{n-1})) a_{nn}^{(-1)} a_{n-1n-1}^{(-1)} (1/(1 + p_{n-2} b_{n-1}))$$
$$= \mathrm{sgn}(b_n a_{n-1} a_{nn}^{(-1)} a_{n-1n-1}^{(-1)}).$$

The last equality and the expression for $\delta(Y_n)$ imply obviously that

$$\mathrm{sgn}(\delta(Y_n)) = -\mathrm{sgn}(a_{n-1n}^{(-1)}/(1 + p_{n-1} b_n)).$$

Further we have

$$Y_{n-1} = p_{n-1} Y_n + Q_{n-1} \text{ if } p_{n-1} \delta(Y_n) \delta(Q_{n-1}) \geq 0.$$

The above expression for Y_{n-1} contains Y_n so that we have to add to the right-hand side conditions the inequality $b_n a_{n-1} a_{nn}^{(-1)} a_{n-1n-1}^{(-1)} \geq 0$. We have

$$\mathrm{sgn}(p_{n-1} \delta(Y_n) \delta(Q_{n-1}))$$
$$= \mathrm{sgn}(p_{n-1} (a_{n-1n}^{(-1)}/(1 + p_{n-1} b_n))(a_{n-1n-1}^{(-1)}/(1 + p_{n-2} b_{n-1})))$$
$$= \mathrm{sgn}(p_{n-1}^2 (a_{nn}^{(-1)}/(1 + p_{n-1} b_n))(a_{n-1n-1}^{(-1)}/(1 + p_{n-2} b_{n-1})))$$
$$= \mathrm{sgn}(a_{n-1n-1}^{(-1)}/(1 + p_{n-2} b_{n-1})).$$

It follows then

$$Y_{n-1} = p_{n-1}Y_n + Q_{n-1} \text{ if } a_{n-1}b_n a_{nn}^{(-1)} a_{n-1n-1}^{(-1)} \geq 0; \tag{7}$$
$$a_{n-1n-1}^{(-1)}/(1 + p_{n-2}b_{n-1}) \geq 0.$$

To prove 2) we apply Theorem 3 for $k = n - 2$. We obtain consecutively

$$Q_{n-1}(\tilde{d}_{n-2}) = (1/(1 + p_{n-2}b_{n-1}))(D_{n-1} - b_{n-1}Q_{n-2}) = Q_{n-1}$$
$$\text{if} \quad a_{n-2}b_{n-1}a_{n-2n-2}^{(-1)} a_{n-1n-1}^{(-1)} \geq 0;$$
$$Y_n(\tilde{d}_{n-2}) = (1/(1 + p_{n-1}b_n))(D_n - b_n Q_{n-1}) = Y_n$$
$$\text{if} \quad a_{n-1}b_n a_{n-1n-1}^{(-1)} a_{nn}^{(-1)} \geq 0$$
$$\text{and } a_{n-2}b_{n-1}a_{n-2n-2}^{(-1)} a_{n-1n-1}^{(-1)} \geq 0;$$
$$Y_{n-1}(\tilde{d}_{n-2}) = (1/(1 + p_{n-1}b_n))Y_n + Q_{n-1} = Y_{n-1}$$
$$\text{if} \quad a_{n-1n-1}^{(-1)}/(1 + p_{n-2}b_{n-1}) \geq 0$$
and the condition by Y_n;
$$Y_{n-2}(\tilde{d}_{n-2}) = p_{n-2}Y_{n-1} + Q_{n-2} = Y_{n-2}$$
$$\text{if} \quad (a_{n-1n-1}^{(-1)}/(1 + p_{n-2}b_{n-1}))(a_{n-2n-2}^{(-1)}/(1 + p_{n-3}b_{n-2}))$$
and the conditions by Y_n,

that is

$$Y_{n-2} = p_{n-2}Y_{n-1} + Q_{n-2} \quad \text{if } a_{n-1}b_n a_{n-1n-1}^{(-1)} a_{nn}^{(-1)} \geq 0,$$
$$a_{n-2}b_{n-1}a_{n-2n-2}^{(-1)} a_{n-1n-1}^{(-1)} \geq 0,$$
$$a_{n-2n-2}^{(-1)}/(1 + p_{n-3}b_{n-2}) \geq 0,$$
$$a_{n-1n-1}^{(-1)}/(1 + p_{n-2}b_{n-1}) \geq 0,$$

which proves 2).

Using (7) we obtain finally

$$Y_{n-2} = p_{n-2}Y_{n-1} + Q_{n-2} \quad \text{if } a_{n-2}b_{n-1}a_{n-2n-2}^{(-1)} a_{n-1n-1}^{(-1)} \geq 0,$$
$$a_{n-2n-2}^{(-1)}/(1 + p_{n-3}b_{n-2}) \geq 0.$$

In the general case we have analogously

$$Y_k = p_k Y_{k+1} + Q_k \quad \text{if } a_k b_{k+1} a_{kk}^{(-1)} a_{k+1k+1}^{(-1)} \geq 0,$$
$$a_{kk}^{(-1)}/(1 + p_{k-1}b_k) \geq 0,$$
$$k = n - 1, n - 2, ..., 1, 0.$$

3. The algorithm with result verification.

Let S be a floating-point system [7], [8] and IS be the set of all intervals with end-points from S. For $a \in R$ denote $\nabla a = \max\{x \in S\colon x \leq a\}$ and $\Delta a = \min\{x \in S\colon x \geq a\}$. The roundings ∇ and Δ generate the computer arithmetic operations (see [7])

$$a \overset{\nabla}{*} b = \nabla(a * b),\ a \overset{\Delta}{*} b = \Delta(a * b), \quad * \in \{+, -, \times, /\}.$$

Let be $A \in IR$, $A = [\underline{a}, \bar{a}]$. Two kinds of monotone roundings for real intervals are used (see [3], [5]):

$$\diamond A = [\nabla \underline{a}, \Delta \bar{a}], \quad \circ A = \begin{cases} [\Delta \underline{a}, \nabla \bar{a}] & \text{if } \Delta \underline{a} \leq \nabla \bar{a}, \\ 0 & \text{otherwise.} \end{cases}$$

They associate the following computer interval arithmetic:

$$A <\!*\!> B = \diamond(A * B), A (*) B = \circ(A * B),$$

where $*$ can be any one of interval-arithmetic operations defined above and $A, B \in IS$.

Let $A, B \in IR$. The following inclusions hold true [3], [5]:

$$\begin{aligned}
&\circ A (+) \circ B \subseteq A + B \subseteq \diamond A <\!+\!> \diamond B;\\
&\circ A (-^{-}) \diamond B \subseteq A -^{-} B \subseteq \diamond A <-^{-}> \diamond B \text{ if } \omega(A) \geq \omega(B),\\
&\diamond A (-^{-}) \circ B \subseteq A -^{-} B \subseteq \circ A <-^{-}> \diamond B \text{ if } \omega(A) < \omega(B);\\
(8) \qquad &\circ A (+^{-}) \diamond B \subseteq A +^{-} B \subseteq \diamond A <\!+\!> \circ B \text{ if } \omega(A) \geq \omega(B),\\
&\diamond A (+^{-}) \circ B \subseteq A +^{-} B \subseteq \diamond A <\!+\!> \diamond B \text{ if } \omega(A) < \omega(B);\\
&\circ A (+) \circ B \subseteq A \times B \subseteq \diamond A <\!+\!> \diamond B;\\
&\circ A (/) \circ B \subseteq A / B \subseteq \diamond A < / > \diamond B.
\end{aligned}$$

Assume now that the entries of the matrix A are machine numbers, i.e. a_i, $b_j \in S$, $i = 0, 1, \ldots, n-1$, $j = 1, 2, \ldots, n$; assume also that the components of the right-hand side interval vector are computer intervals, i.e. $D_i in IS$ for $i = 0, 1, \ldots, n$. Suppose further that inclusions for the elements $a_{ij}^{(-1)}$ of the inverse matrix are known and denote by ε_{ij} their signs, i.e. $\varepsilon_{ij} = \text{sgn} a_{ij}^{(-1)}$.

We formulate now the algorithm with result verification $(CA1) - (CA2)$, related to $(A1) - (A2)$.

(CA1).

$\bigcirc p_0 = -\diamond p_0 = [-a_0, a_0]$;

For $i = 1$ to $n-1$ compute

$$\diamond(1+p_{i-1}b_i) = 1 <+> \diamond(p_{i-1} <\times> b_i)$$
$$\bigcirc(1+p_{i-1}b_i) = 1(+)\nabla(p_{i-1}b_i) = 1(+)\bigcirc p_{i-1}(\times)b_i;$$
$$\diamond p_i = -\diamond(a_i/(1+p_{i-1}b_i)) = -a_i </> \diamond(1+p_{i-1}b_i);$$
$$\bigcirc p_i = -\bigcirc(a_i/(1+p_{i-1}b_i)) = -a_i(/)\bigcirc(1+p_{i-1}b_i);$$

end;

$\bigcirc Q_0 \diamond Q_0 = D_0$;

For $i = 1$ to $n-1$ compute

$$\bigcirc Q_i = \bigcirc(1/(1+p_{i-1}b_i))(\times)(D_i(-^{-})b_i(\times)\bigcirc Q_{i-1});$$
$$\diamond Q_i = \diamond(1/(1+p_{i-1}b_i)) {<\!\!\times\!\!>} (D_i <-^{-}> b_i {<\!\!\times\!\!>} \diamond Q_{i-1});$$
$$\mathrm{sgn}(\delta(Q_i)) = -\mathrm{sgn}(\varepsilon_{ii}/\diamond(1+p_{i-1}b_i));$$

end;

(CA2).

Compute $Y_n = \diamond(1/(1+p_{n-1}b_n)) {<\!\!\times\!\!>} (D_n <-^{-}> b_n {<\!\!\times\!\!>} \diamond Q_{n-1})$;

For $k = n-1$ downto 0 do begin

For $i = 0$ to n determine $\delta(D_i) = -\varepsilon_{ki}\omega(D_i)$; end;

For $i = k+1$ to $n-1$ compute

$$\bigcirc Q_i(\tilde{d}_k) = \begin{cases} \left.\begin{array}{r} \bigcirc(1/(1+p_{i-1}b_i))(\times)(D_i(-^{-})b_i {<\!\!\times\!\!>} \diamond Q_{i-1}(\tilde{d}_k)) \\ \text{if } \omega(D_i) \ge |b_i\delta(Q_i(\tilde{d}_k))|, \\ \bigcirc(1/(1+p_{i-1}b_i))(\times)(D_i(-^{-})b_i(\times)\bigcirc Q_{i-1}(\tilde{d}_k)) \\ \text{if } \omega(D_i) < |b_i\delta(Q_i(\tilde{d}_k))|, \end{array}\right\} b_i\varepsilon_{ki}\delta(Q_{i-1}(\tilde{d}_k)) \le 0, \\ \bigcirc(1/(1+p_{i-1}b_i))(\times)(D_i(-^{-})b_i(\times)\bigcirc Q_{i-1}(\tilde{d}_k)) \text{ if } b_i\varepsilon_{ki}\delta(Q_i(\tilde{d}_k)) > 0; \end{cases}$$

$$\diamond Q_i(\tilde{d}_k) = \begin{cases} \left.\begin{array}{r} \diamond(1/(1+p_{i-1}b_i)) {<\!\!\times\!\!>} (D_i <-^{-}> b_i(\times)\bigcirc Q_{i-1}(\tilde{d}_k)) \\ \text{if } \omega(D_i) \ge |b_i\delta(Q_i(\tilde{d}_k))|, \\ \diamond(1/(1+p_{i-1}b_i)) {<\!\!\times\!\!>} (D_i(-^{-})b_i {<\!\!\times\!\!>} \diamond Q_{i-1}(\tilde{d}_k)) \\ \text{if } \omega(D_i) < |b_i\delta(Q_i(\tilde{d}_k))|, \end{array}\right\} b_i\varepsilon_{ki}\delta(Q_{i-1}(\tilde{d}_k)) \le 0, \\ \diamond(1/(1+p_{i-1}b_i)) {<\!\!\times\!\!>} (D_i <-^{-}> b_i(\times)\diamond Q_{i-1}(\tilde{d}_k)) \text{ if } b_i\varepsilon_{ki}\delta(Q_i(\tilde{d}_k)) > 0; \end{cases}$$

$\delta(Q_i(\tilde{d}_k)) = \diamond(1/(1+p_{i-1}b_i)){<}{\diamond}{>}(\delta(D_i) <-> b_i{<}{\diamond}{>}\delta(Q_{i-1}(\tilde{d}_k));$

end;

$\{Q_k(\tilde{d}_k) = Q_k\};$

Compute

$$\bigcirc Y_n(\tilde{d}_k) = \begin{cases} \left.\begin{array}{r} \bigcirc(1/(1+p_{n-1}b_n))(\times)(D_n(-^-)b_n{<}{\diamond}{>}\diamond Q_{n-1}(\tilde{d}_k)) \\ \text{if } \omega(D_n) \geq |b_n\delta(Q_n(\tilde{d}_k))|, \\ \bigcirc(1/(1+p_{n-1}b_n))(\times)(D_n(-^-)b_n(\times)\bigcirc Q_{n-1}(\tilde{d}_k)) \\ \text{if } \omega(D_n) < |b_n\delta(Q_n(\tilde{d}_k))|, \end{array}\right\} b_n\varepsilon_{kn}\delta(Q_{n-1}(\tilde{d}_k)) \leq 0, \\ \bigcirc(1/(1+p_{n-1}b_n))(\times)(D_n(-^-)b_n(\times)\bigcirc Q_{n-1}(\tilde{d}_k)) \text{ if } b_n\varepsilon_{kn}\delta(Q_n(\tilde{d}_k)) > 0; \end{cases}$$

$$\diamond Y_n(\tilde{d}_k) = \begin{cases} \left.\begin{array}{r} \diamond(1/(1+p_{n-1}b_n)){<}{\diamond}{>}(D_n <-^-> b_n(\times)\bigcirc Q_{n-1}(\tilde{d}_k)) \\ \text{if } \omega(D_n) \geq |b_n\delta(Q_n(\tilde{d}_k))|, \\ \diamond(1/(1+p_{n-1}b_n)){<}{\diamond}{>}(D_n(-^-)b_n{<}{\diamond}{>}\diamond Q_{n-1}(\tilde{d}_k)) \\ \text{if } \omega(D_n) < |b_n\delta(Q_n(\tilde{d}_k))|, \end{array}\right\} b_n\varepsilon_{kn}\delta(Q_{n-1}(\tilde{d}_k)) \leq 0, \\ \diamond(1/(1+p_{n-1}b_n)){<}{\diamond}{>}(D_n <-^-> b_n(\times)\diamond Q_{n-1}(\tilde{d}_k))) \text{ if } b_n\varepsilon_{kn}\delta(Q_n(\tilde{d}_k)) > 0; \end{cases}$$

$\delta(Q_n(\tilde{d}_k)) = \diamond(1/(1+p_{n-1}b_n)){<}{\diamond}{>}(\delta(D_n) <-> b_n{<}{\diamond}{>}\delta(Q_{n-1}(\tilde{d}_k));$

For $i = n-1$ **downto** $k+1$ **compute**

$$\bigcirc Y_i(\tilde{d}_k) = \begin{cases} \bigcirc p_i(\times)\bigcirc Y_{i+1}(\tilde{d}_k)(+)\bigcirc Q_i(\tilde{d}_k) \text{ if } p_i\delta(Y_{i+1}(\tilde{d}_k))\delta(Q_i(\tilde{d}_k)) \geq 0; \\ \left.\begin{array}{l} \bigcirc p_i(\times)\bigcirc Y_{i+1}(\tilde{d}_k)(+^-)\diamond Q_i(\tilde{d}_k) \\ \quad \text{if } |p_i\delta(Y_{i+1}(\tilde{d}_k))| \geq |\delta(Q_i(\tilde{d}_k))|, \\ \diamond p_i{<}{\diamond}{>}\diamond Y_{i+1}(\tilde{d}_k)(+^-)\bigcirc Q_i(\tilde{d}_k) \\ \quad \text{if } |p_i\delta(Y_{i+1}(\tilde{d}_k))| < |\delta(Q_i(\tilde{d}_k))|, \end{array}\right\} p_i\delta(Y_{i+1}(\tilde{d}_K))\delta(Q_i(\tilde{d}_k)) < 0; \end{cases}$$

$\diamond Y_i(\tilde{d}_k)$

$$= \begin{cases} \diamond p_i \mathbin{<\!\!\times\!\!>} \diamond Y_{i+1}(\tilde{d}_k) \mathbin{<\!\!+\!\!>} \diamond Q_i(\tilde{d}_k) \text{ if } p_i\delta(Y_{i+1}(\tilde{d}_k))\delta(Q_i(\tilde{d}_k)) \geq 0; \\ \left.\begin{array}{l} \circ p_i \mathbin{<\!\!\times\!\!>} \diamond Y_{i+1}(\tilde{d}_k) \mathbin{<\!\!+\!\!>} \circ Q_i(\tilde{d}_k) \\ \quad \text{if } |p_i\delta(Y_{i+1}(\tilde{d}_k))| \geq |\delta(Q_i(\tilde{d}_k))|, \\ \diamond p_i (\times) \circ Y_{i+1}(\tilde{d}_k) \mathbin{<\!\!\times\!\!>} \diamond Q_i(\tilde{d}_k) \\ \quad \text{if } |p_i\delta(Y_{i+1}(\tilde{d}_k))| < |\delta(Q_i(\tilde{d}_k))|, \end{array}\right\} p_i\delta(Y_{i+1}(\tilde{d}_k))\delta(Q_i(\tilde{d}_k)) < 0; \end{cases}$$

$\delta(Y_i(\tilde{d}_k)) = p_i\delta(Y_{i+1}(\tilde{d}_k)) + \delta(Q_i(\tilde{d}_k))$;

end;

$$Y_k = \begin{cases} \diamond p_k \mathbin{<\!\!\times\!\!>} \diamond Y_{k+1}(\tilde{d}_k) \mathbin{<\!\!+\!\!>} \diamond Q_k \text{ if } a_k\varepsilon_{kk}\delta(Y_{k+1}(\tilde{d}_k)) \geq 0; \\ \left.\begin{array}{l} \diamond p_k \mathbin{<\!\!\times\!\!>} \diamond Y_{k+1}(\tilde{d}_k) \mathbin{<\!\!+\!\!>} \circ Q_k \\ \quad \text{if } |p_k\delta(Y_{k+1}(\tilde{d}_k))| \geq |\delta(Q_k)|, \\ \circ p_k (\times) \circ Y_{k+1}(\tilde{d}_k) \mathbin{<\!\!\times\!\!>} \diamond Q_k \\ \quad \text{if } |p_k\delta(Y_{k+1}(\tilde{d}_k))| < |\delta(Q_k)|, \end{array}\right\} a_k\varepsilon_{kk}\delta(Y_{k+1}(\tilde{d}_k)) < 0; \end{cases}$$

end.

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Institute of Biophysics
Bulgarian Academy of Sciences
Acad. G. Bonchev str. bl.23
1113 Sofia,
BULGARIA

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