Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New series Vol.8, 1994, Fasc.2–3

On HB-Parallel Hyperbolic Kaehlerian Spaces

Nevena Pušić

Presented by D. Kurepa

In this paper a classification of hyperbolic Kaehlerian spaces which have parallel Bochner curvature tensor of hyperbolic Kaehlerian space (HB-tensor) is given. Also, there is given a classification of HB-parallel hyperbolic Kaehlerian spaces with a recurrent Ricci tensor.

Preliminaries

If an $n(=2m)$ - dimensional pseudo Riemannian space Mn with metric g_{ij} is equipped with a non-degenerate structure tensor F_j^i which satisfies the following conditions:

(0.1) $$F_{jk}{}^{i} = 0$$

(0.2) $$F_j{}^i F_i{}^k = \delta_j{}^k$$

(0.3) $$F_{jk} = F_j{}^i g_{ik} = -F_{kj}$$

then the space Mn is called hyperbolic Kaehlerian space.

Hyperbolic Kaehlerian spaces have been introduced in 1948 by P.K. Rashevski [14], but by use of a special coordinate system. A.P. Shirokov [17] enabled to use an arbitrary real coordinate system, involving the structure tensor. Later a number of authors have been working on this topic. Here we will mention just some of them: G. L. Bejan ([3],[4]), A. Borisov and G. Ganchev [7], G. Djelepov [6], G. Djelepov and G. Markov [9], N. Barros and A. Romero [2], E. Pavlov [12].

The subject of this paper is very special, so we shall not use the results of these authors directly.

As we proved in the paper [13], the non-degenerate structure possesses n (the dimension of the space $n = 2m$) linearly independent eigenvectors in the tangent space. In the paper [13], we also proved

Proposition 1. *(A) Every vector in the tangent space of a hyperbolic Kaehlerian space is transformed by the structure tensor into an orthogonal vector*
(B) The scalar square of a vector-original is opposite to the scalar square of the vector image. ∎

In accordance to Proposition 1. eigen vectors of the structure are isotropic (null-vectors). As the space possesses n linearly independent eigen vectors, there exists a basis of the tangent space of a hyperbolic Kaehlerian space where these isotropic vectors serve as a basic vector fields. In such a basis metric tensor is hybrid and the structure tensor is pure. Covariant structure tensor is also hybrid. Using this coordinate system, we can show that a hyperbolic Kaehlerian space admits isotropic vector fields which are not eigen for the structure. Moreover, such a coordinate system shows that a hyperbolic Kaehlerian space is divided very naturally into two totally geodesic subspaces of equal dimension. Such a basis is called separated basis. Also according to Proposition 1(B), there exists vectors if positive scalar square (space-like vectors) and vectors of negative scalar square (time-like vectors). Space-like vectors may serve as a domain for the involution $F_j^{\ i}$ and its co-domain will be the set of the time-like vectors. We may choose such a basis; then the metric tensor will be a pure tensor of signature (m, m) and $(F_j^{\ i})$ will be a hybrid tensor. Such a basis is called an adapted basis.

For all the considerations in this paper we will use an arbitrary chosen basis- it will be neither separated nor adapted. However, all our results can be transferred into these special bases and some of them may even look simpler.

In the article [13], we have investigated the properties of a hyperbolic Kaehlerian space. Among other properties, we investigated a conformal connection (as there cannot be a conformal transformation naturally introduced) and we found a tensor which is an invariant for all conformal connections on a hyperbolic Kaehlerian space

$$
\begin{aligned}
(0.4)\qquad HB^i{}_{jkl} = HB^i{}_{jkl} &- \frac{1}{n+4}[\,\delta_l{}^i K_{kj} - \delta_k{}^i K_{lj} + g_{kj} K_l{}^i - g_{lj} K^i_k \\
&+ F_l{}^i S_{kj} - F_k{}^i S_{lj} + F_{kj} S_l{}^i - F_{lj} S_k{}^i + 2S_j{}^i F_{kl} + S_k{}^l F_{ji} \\
&- \frac{K}{n+2}(\delta_l{}^i g_{kj} - \delta_k{}^i g_{lj} + F_{lj} F_k{}^i - F_l{}^i F_{kj} - 2F_j{}^i F_{kl})
\end{aligned}
$$

By K^i_{jkl} we denote a curvature tensor of Levi-Civita connection for the metric (g_{ij}), by K_{kj} the corresponding Ricci tensor and by K the corresponding scalar curvature. Also, there holds:

$$
(0.5)\qquad S_{ij} = K_{la} F_j{}^a
$$

In the paper [13] we proved that the tensor S_{ij} is skew symmetric. The tensor HB is a curvature-like tensor and in [6] we proved the following algebraic properties of this tensor:

$$
(0.6)\qquad
\begin{aligned}
&(a)\ HB_{ijkl} = -HB_{ijlk} \\
&(b)\ HB_{ijkl} = -HB_{jikl} \\
&(c)\ HB_{ijkl} = HB_{klij} \\
&(d)\ HB_{ijkl} + HB_{iklj} + HB_{iljk} = 0 \\
&(e)\ HB^t{}_{jkt} = 0 \\
&(f)\ HB^i{}_{tkl} F_j{}^t - HB^t{}_{jkl} F_t{}^i = 0
\end{aligned}
$$

We call the tensor HB a Bochner curvature of a hyperbolic Kaehlerian space, for the sake of two analogies: it looks like Bochner tensor (of a Kaehlerian space) and the Bochner curvature tensor is, in some sense, an invariant tensor of conformal connections in a Kaehlerian space–if this space is flat (K. Yano [18]).

T. Adati and T. Miyazawa and W. Roter with his group investigated properties of conformally symmetric pseudo Riemannian spaces, particularly if they are Ricci-recurrent ([1],[15]). On the other hand, M. Matsumoto ([10]), M. Matsumoto and S. Tanno ([11]), B. Chen and K. Yano ([5]), Y. Kubo ([8]) and other authors investigated Kaehlerian spaces with parallel or vanishing Bochner tensor. These investigations gave the motivation to this author to try to obtain more or less analogous results for hyperbolic Kaehlerian spaces.

§1. Classification of HB-parallel hyperbolic Kaehlerian spaces

As we going to differentiate the HB-tensor covariantly, we have to construct a covariant derivative for the Ricci tensor first.

In a hyperbolic Kaehlerian space the following formulae are valid:

$$K^a{}_{jkl,a} = K_{jk,l} - K_{jl,k} \; ; \; K_{,l} = 2K_l{}^a{}_{,a} \tag{1.1}$$

$$F_j{}^a K_{ak} = -K_{ak} F_k{}^a \; ; \; F_j{}^a K_a{}^k = -K_j{}^a F_a{}^k \tag{1.2}$$

$$K^i{}_{jkl} F_m^{\ l} = K^i_{jlm} F_k{}^l \; ; \; K^i{}_{jkl} F_m^{\ i} = K^j_{mkl} F_j{}^i \tag{1.3}$$

where $K_{,k}$ stands for the partial derivative of the scalar curvative function.

From (1.1) and (0.5) we can obtain:

$$2S_k{}^a{}_{,a} = -K_{,b} F_k{}^b \tag{1.4}$$

Using (1.3) in some changed form, we obtain:

$$S_{ij} = K_{pijq} F^{qp}$$

and, using this,

$$S_{ij} = -\frac{1}{2} F^{sm} K_{msij} \tag{1.5}$$

by the first Bianchi identity. Using the first Bianchi identity again

$$K_{ijkl,s} + K_{ijls,k} + K_{ijsk,l} = 0$$

and transvecting it by F^{ij}, taking to account (1.5) and omitting the factor $\frac{1}{2}$ we obtain

$$S_{kl,s} + S_{ls,k} + S_{sk,l} = 0$$

and, transvecting again by F_j^s

$$F_j{}^s S_{kl,s} = K_{kj,l} - K_{lj,k}$$

After transvection by $F_b^j \; F_a^j$, we finally obtain

$$K_{kb,a} = F_b{}^l F_a{}^j (K_{kj,l} - K_{lj,k}) \tag{1.6}$$

Our hyperbolic Kaehlerian space is HB-parallel. Differentiating covariantly HB-tensor $_{,m}$ contracting the indices i and m and taking into account (1.1), we obtain

$$K_{jk,l} - K_{jl,k} = \frac{1}{n+4}[4(K_{kj,l} - K_{lj,k}) + \frac{n}{2(n+2)}(K_{,l}g_{kj} - K_{,k}g_{lj} + K_{,a}F_k{}^aF_{lj} - K_{,a}F_l{}^aF_{kj} - 2K_{,a}F_j{}^aF_{kl})$$

Dividing by $\frac{n}{n+4}$, we obtain

$$K_{kj,l} - K_{lj,k} = \frac{1}{2(n+2)}(K_{,l}g_{kj} - K_{,k}g_{lj} + K_{,a}F_k{}^aF_{lj} - K_{,a}F_l{}^aF_{kj} - 2K_{,a}F_j{}^aF_{kl}) \tag{1.7}$$

and using (1.6)

$$K_{kj,l} = \frac{1}{2(n+2)}K_{,k}g_{jl} + K_{,j}g_{kl} + K_{,a}F_j{}^aF_{lk} + K_{,a}F_k{}^aF_{lj} + 2K_{,l}g_{kj}) \tag{1.8}$$

As we have supposed, the HB-tensor is parallel. Using the Ricci identity we obtain

$$K^{\,a}_{mkl}HB_{astp} + K^{\,a}_{skl}HB_{matp} + K^{\,a}_{tkl}HB_{msap} + K^{\,a}_{pkl}HB_{msta} = 0$$

Taking into account the algebraic properties of the curvative tensor and (0.6), we can re-write the last equality in this way:

$$K_{lkma}HB^a{}_{stp} - K_{lksa}HB^a{}_{mtp} + K_{lkta}HB^a{}_{pms} - K_{lkpa}HB^a{}_{tms} = 0$$

Operating $\overset{l}{;}$ on this formula and using (1.1), we obtain

$$(K_{km,a} - K_{ka,m})HB^a{}_{stp} - (K_{ks,a} - K_{ka,s})HB^a{}_{mtp} + (K_{kt,a} - K_{ka,t})HB^a{}_{pms} - (K_{kp,a} - K_{ka,p})HB^a{}_{tms} = 0$$

Using the formula (1.7) and (0.6)(f) or (1.3) we obtain

$$\begin{aligned} &K_{,}{}^b(F_{am}F_{sk}HB^a{}_{btp} - F_{as}F_{mk}HB^a{}_{btp} \\ &+F_{at}F_{pk}HB^a{}_{bms} - F_{ap}F_{tk}HB^a{}_{bms}) \\ &-K_{,m}HB_{kstp} - K_{,s}HB_{kmtp} - K_{,t}HB_{kpms} - K_{,p}HB_{ktms} \\ &+K_{,b}(F_m{}^bF_{as}HB^a{}_{ktp} - F_s{}^bF_{am}HB^a{}_{ktp} \\ &+F_t{}^bF_{ap}HB^a{}_{kms} - F_p{}^bF_{at}HB^a{}_{kms}) \\ &+K_{,a}(HB^a{}_{stp}g_{mk} - HB^a{}_{mtp}g_{sk} + HB^a{}_{pms}g_{kt} - HB^a{}_{tms}g_{pk}) = 0 \end{aligned} \tag{1.9}$$

Using the identities

$$F_{ab} HB_{ajkb} = 0 \tag{1.10}$$

$$F_i{}^a F_j{}^b HB_{abkl} = -HB_{ijkl} \tag{1.11}$$

which are easy to prove, (0.6)(d) and (0.6)(f) or (1.3), transvecting (1.9) by g^{mk}, we obtain

$$(n+2)K_{,a} HB^a{}_{stp} = 0 \tag{1.12}$$

Transvecting now (1.9) by $K_{,m}$ we obtain

$$K_{,m} K_{,}{}^m HB_{kstp} = 0 \tag{1.13}$$

There are three possibilities for (1.13)

$$K_{,m} = 0 \tag{1.14}$$

or

$$HB_{kstp} = 0 \tag{1.15}$$

or

$$K_{,m} K_{,}{}^m = 0 \tag{1.16}$$

If (1.14) holds, then, according to (1.8) the Ricci tensor is parallel and also the tensor S_{ij} is parallel. As the space is HB-parallel, it has to be a symmetric space in the sense of Cartan.

If (1.15) holds, then the HB-tensor vanishes.

We have proved

Theorem 1.1. *If a hyperbolic Kaehlerian space is HB-parallel, then one of the following cases occurs:*

[(i)] The scalar curvature is constant and the space is symmetric in the sense of Càrtan

[(ii)] The tensor HB vanishes

[(iii)] The gradient of the scalar curvative is an isotropic vector field (i.e. the scalar curvative function satisfies the partial differential equation Δ_j K) and $K_{,a} HB^a_{stp} = 0$ ■.

§2. Some considerations of essentially HB-parallel hyperbolic Kaehlerian spaces

If a hyperbolic Kaehlerian space is essentialy HB-parallel (case (iii) of the Theorem 1.1). Then $K_{,i}$ is an isotropic vector field and

$$K_{,a}HB^{a}{}_{stp} = 0 \tag{2.1}$$

$K_{,i}$ is a gradient vector field. If, moreover, $K_{,i}$ is a parallel vector field, then

$$K_{,a}K^{a}{}_{stp} = 0 \tag{2.2}$$

by the Ricci identity and, consequently,

$$K_{,a}K_{p}{}^{a} = 0 \tag{2.3}$$

Using the formula (1.8) we can find

$$(K_{kj}K^{kj})_{,l} = \frac{2K}{n+2}K_{,l} \tag{2.4}$$

and

$$K^{lk}K_{kj,l} = \frac{K}{2(n+2)}K_{,j} = \frac{1}{4}(K_{lk}K^{lk})_{,j} \tag{2.5}$$

As there holds

$$K^{kj,l} = \frac{1}{2(n+2)}(K_{,a}F^{ja}F^{lk} + K_{,a}F^{ka}F^{lj} + K^{,k}g^{jl} + K^{,j}g^{kl} + 2K^{,l}g^{kj})$$

then

$$K_{lksj}K^{kj,l} = 0 \tag{2.6}$$

Using the formula (1.8) and the Ricci identity again

$$K_{kj,l,s} - K_{kj,s,l} = -K^{a}{}_{kls}K_{aj} - K^{a}{}_{jls}K_{ka}$$

we obtain

$$\begin{aligned}-K^{a}{}_{kls}K_{aj} - K^{a}{}_{jls}K_{ka} = \frac{1}{2(n+2)}(&K_{,a,s}F_{j}{}^{a}F_{lk} + K_{,a,s}F_{k}{}^{a}F_{lj} \\ &+ K_{,k,s}g_{jl} + K_{,j,s}g_{kl} \\ &- K_{,a,l}F_{j}{}^{a}F_{sk} + K_{,a,l}F_{k}{}^{a}F_{lj} \\ &+ K_{,k,l}g_{js} + K_{,j,l}g_{ks}\end{aligned}$$

After transformation by g^{kl} there holds

$$-K_s{}^a K_{aj} - K_{ajls}K^{la} = \frac{1}{2(n+2)}[(n-1)K_{,j,s} - K_{,a,l}F_j{}^a F_{sl} - K_{,a}{}^{,a} g_{js}]$$

Now we operate ^{,s} on this equality and then

$$\frac{1}{2(n+2)}[(n-2)K_{j,s}{}^{,s} - K_{,a,l}{}^{,s}F_j{}^a F_{sl}] = 0$$

The last member of the left-hand side of this equality we can write

$$K_{,a,l,s}F_j{}^a F^{sl} = - K_{,a,s,l}F_j{}^a F^{ls} = 0$$

and finally

$$K_{,j,s}{}^{,s} = K_{,s}{}^{,s}{}_{,j} = 0 \tag{2.7}$$

From (2.7) there yields

$$K_{,s}{}^{,s} = const \tag{2.8}$$

for the scalar function $K_{,s}{}^{,s}$

We have proved

Theorem 2.1. *If a hyperbolic Kaehlerian space is essentially HB-parallel, then the gradient vector of the scalar curvature is a null (isotropic) vector, i.e.*

$$\Delta_1 K = 0$$

where Δ_1 denotes the first Beltrami differential operator and K is the scalar curvature function. If $K_{,a}$ is a parallel vector field, there also holds

$$\Delta_2 K = 0 \tag{2.9}$$

where Δ_2 denotes the second Beltrami differential operator (Laplace transform) i.e.

$$\Delta_2 = {}_{,a,b}g^{ab} \quad \blacksquare$$

§3. Ricci-recurrent HB-parallel hyperbolic Kaehlerian spaces

A metric space is said to be Ricci-recurrent if its Ricci tensor satisfies the relation

$$K_{ij,k} = \chi_k K_{ij} \tag{3.1}$$

where (χ_k) are components of some vector field.

In this paragraph we shall mainly use the method and the notation of T. Adati and T. Miyazawa ([1]), wherever is possible.

In order to avoid very long and complicated calculations, we introduce some abbreviations

$$\mathrm{II}_{kj} = \frac{1}{(n+4)}\left[K_{kj} - \frac{K}{2(n+2)}g_{kj}\right] \quad , \mathrm{II}_{kj} = \mathrm{II}_{jk} \tag{3.2}$$

$$T_{kj} = \frac{1}{(n+4)}\left[S_{kj} - \frac{K}{2(n+2)}F_{kj}\right] \quad , T_{kj} = -T_{jk} \tag{3.3}$$

Also there holds

$$T_{kj} = \mathrm{II}_{ka}F_j{}^a \tag{3.4}$$

We can put

$$HB^i{}_{jkl} = K^i{}_{jkl} - D^i{}_{jkl} \tag{3.5}$$

where

$$\begin{aligned} D^i{}_{jkl} = {} & \delta_l{}^i \mathrm{II}_{kj} - \delta_k{}^i \mathrm{II}_{lj} + g_{kj}\mathrm{II}_l{}^i - g_{lj}\mathrm{II}_k{}^i \\ & + F_l{}^j T_{ki} - F_k{}^j T_{lj} + F_{kj}T_l{}^i - F_{lj}T_k{}^i \\ & + 2T_j{}^i F_{kl} + 2T_{kl}F_j{}^i \end{aligned} \tag{3.6}$$

II_{ij} and $D^i{}_{jkl}$ satisfy the following relations

$$\mathrm{II} = g^{ab}\mathrm{II}_{ab} = \frac{K}{2(n+2)} \tag{3.7}$$

$$D_{ijkl} = g_{ia}D^a{}_{jkl} = -D_{jikl} = -D_{ijlk} = D_{klij} = D_{lkji} \tag{3.8}$$

If HB-parallel hyperbolic Kaehlerian space is Ricci-recurrent, we shall first prove

Lemma 3.1. *K_{ij} is Ricci-recurrent if and only if II_{ij} is recurrent*

Proof. : If we put

$$\mathrm{II}_{ij,k} = \chi_k \mathrm{II}_{ij} \tag{3.9}$$

Contracting by metric tensor we can get

$$\mathrm{II}_{,l} = \mathrm{II}\chi_l \text{ or according to (3.7) } K_{,l} = K\chi_l \tag{3.10}$$

Then, according to (3.2) yields

$$K_{ij,l} = \chi_l K_{ij} \tag{3.11}$$

Conversely, if (3.11) holds, then according to (3.2) and (3.7) holds (3.10) and also holds (3.9) ∎

There also holds the following

Lemma 3.2. *If in a HB-parallel hyperbolic Kaehlerian space the Ricci tensor is recurrent, then the tensor T_{kj} is recurrent with the same recurrence vector.* ∎

It is easy to prove Lemma 3.2 using Lemma 3.1 and (3.3).

The following two Lemmas were proved by A. G. Walker and we shall give them without proof:

Lemma 3.3. *(A. G. Walker) The curvature tensor of a Riemannian space satisfies the identity*

$$\begin{aligned} &R_{ijkl,m,h} - R_{ijkl,h,m} + R_{klmh,i,j} \\ &-R_{klmh,j,i} + R_{mhij,k,l} - R_{mhij,l,k} = 0 \ \blacksquare \end{aligned} \tag{3.12}$$

Lemma 3.2. *(A. G. Walker) If $a_{\alpha\beta}$ and b_α are numbers satisfying*

$$a_{\alpha\beta} = a_{\beta\alpha}, \quad a_{\alpha\beta}b_\gamma + a_{\beta\gamma}b_\alpha + a_{\gamma\alpha}b_\beta = 0 \tag{3.13}$$

where α, $,\beta$, $\gamma = 1, 2, \ldots, N$, the all $a_{\alpha\beta}$ vanish or all b_α vanish.∎Using this Lemmas we can prove

Theorem 3.1. *In a HB-parallel hyperbolic Kaehlerian space, if $K_{ij,l} = \chi_l K_{ij}$ for a non-zero vector χ_l and nonzero tensor K_{ij}, then the recurrence vector is gradient.*

Proof. If we suppose $K_{ij,l} = \chi_l K_{ij}$ and according to Lemma 1.1 and 1.2 $\mathrm{II}_{ij,l} = \chi_l \mathrm{II}_{ij}$ and $T_{ij,l} = \chi_l T_{ij}$, then

$$D^i{}_{jkl,m} = \chi_m D^i{}_{jkl} \tag{3.14}$$

because the metric tensor and structure tensor are covariantly constant.

As the space is HB-parallel, then according to (3.5)

$$K^i{}_{jkl,m} = \chi_m D^i{}_{jkl} \tag{3.15}$$

Differentiating (3.15) covariantly and using (3.14), we obtain

$$K^i{}_{jkl,m,h} = \chi_{m,h} D^i{}_{jkl} + \chi_h \chi_m D^i{}_{jkl}$$

what means

$$K_{ijkl,m,h} = \chi_{mh} D_{ijkl}$$

where we put $\chi_{mh} = \chi_{m,h} - \chi_{h,m}$.

According to (3.12), we obtain

$$\chi_{mh} D_{ijkl} + \chi_{ij} D_{klmh} + \chi_{kl} D_{mhij} = 0 \tag{3.16}$$

Now we shall substitute pairs of indices m, h; i, j; k, l; respectively for α, β, γ, χ_{mh}, we shall denote by b_α etc., and D_{ijkl} by $a_{\alpha\beta}$ etc. According to (3.8) it will be $a_{\beta\gamma} = a_{\gamma\beta}$. (3.8) and (3.16) represent the relation (3.13) and Lemma (3.4). We have two possibilities: $\chi_{mh} = 0$ (for every m and h) or $D_{ijkl} = 0$. As $K_{ij} \neq 0$, then $\mathrm{II}_{ij} \neq 0$and there cannot be $D_{ijkl} = 0$ for every set of indices. Then χ_{mh} vanishes and, consequently (χ_{m}) is a gradient. ■

§4. Classification of Ricci-recurrent HB-parallel hyperbolic Kaehlerian spaces

It is easy to prove the following

Theorem 4.1. *HB-parallel hyperbolic Kaehlerian space which has vanishing Ricci tensor is symmetric in the sense of Certan* ■

If $K_{ij} \neq 0$, then $K = 0$ and $\mathrm{II}_{ij} = 0$ and $T_{ij} = 0$ and $D_{ijkl} = 0$. According to (3.5) it will be $HB_{ijkl} = K_{ijkl}$ and then K_{ijkl} is parallel and the space is symmetric.

Suppose, now, that $K_{ij} \neq 0$. It easy to see that then $\mathbf{II}_{ij} \neq 0$. As the space is Ricci recurrent, then, by Lemma 3.1, $\mathbf{II}_{ij,k} = \chi_l \mathbf{II}_{ij}$.

We shall prove the following

Lemma 4.1. *For a HB-parallel Ricci-recurrent hyperbolic Kaehlerian space the following relations are fulfilled:*

$$HB^t{}_{jmh}II_{tk} + HB^t{}_{kmh}II_{jt} = 0 \quad \text{and} \quad II^{tm}HB_{tjmh} = 0$$
$$HB^t{}_{jmh}T_{tk} + HB^t{}_{kmh}T_{jt} = 0 \quad \text{and} \quad T^{tm}HB_{tjmh} = 0$$
$$HB^t{}_{jmh}F_{tk} + HB^t{}_{kmh}F_{jt} = 0 \quad \text{and} \quad F^{tm}HB_{tjmh} = 0$$

We could add the fourth relation, for the metric tensor, but it would give only (0.6)(a). Also the relation (4.3) is in fact, unnecessary because it is variant of (0.6)(f), but we shall use it in very same form.

Proof. As $\mathbf{II}_{ij,k} = \chi_l \mathbf{II}_{ij}$. and χ_l is a gradient, then there holds

$$\mathbf{II}_{ij,l,m} = \mathbf{II}_{ij,l,m} \tag{4.1}$$

If we apply to (4.4) the Ricci identity, using (3.5), then we obtain

$$\mathbf{II}_{tj}HB^t{}_{iml} + \mathbf{II}_{tj}D^t{}_{iml} + \mathbf{II}_{it}HB^t{}_{jml} + \mathbf{II}_{it}D^t{}_{jml} = 0 \tag{4.2}$$

Now we differentiate covariantly (4.5) and use facts that HB-tensor is parallel and tensors $\mathbf{II}_{ij}$ and D_{ijkl} are recurrent with recurrence vector (χ_l). We obtain

$$\chi_s(\mathbf{II}_{tj}HB^t{}_{iml} + \mathbf{II}_{it}HB^t{}_{jml}) + 2\chi_s(\mathbf{II}_{tj}D^t{}_{iml} + \mathbf{II}_{it}HB^t{}_{jml}) = 0$$

As $\chi_s \neq 0$ for at least one s, then

$$\mathbf{II}_{tj}HB^t{}_{iml} + \mathbf{II}_{it}HB^t{}_{jml} + 2(\mathbf{II}_{tj}D^t{}_{iml} + \mathbf{II}_{it}D^t{}_{jml}) = 0 \tag{4.3}$$

Comparing (4.6) with (4.5), we obtain

$$\mathbf{II}_{tj}HB^t{}_{iml} + \mathbf{II}_{it}HB^t{}_{jml}$$

and

$$\mathbf{II}_{tj}D^t{}_{iml} + \mathbf{II}_{it}D^t{}_{jml}$$

First one of these two equalities is the result (4.1), if one changes places of indices in a proper way. If we transvect it by g_{im}, then we get:

$$\mathbf{II}^{tm}HB_{tjml} + \mathbf{II}^t{}_jHB_{timl}g^{im} = 0$$

As the contraction of HB-tensor in the upper and any of lower indices gives zero, then there yields:

$$\mathrm{II}^{tm} HB_{tjml} = 0$$

which is the second part of the relation (4.1).

The relation (4.2) can be proved in the exactly same way, but the Lemma 3.2 has to be used first. ■

As the space is HB-parallel, then

$$HB^h{}_{ijk,l,m} - HB^h{}_{ijk,m,l} = 0$$

Applying the Ricci identity and using (3.5), we obtain

$$\begin{aligned} & HB^h{}_{tlm} HB^t{}_{ijk} - HB^t{}_{ilm} HB^h{}_{tjk} - HB^t{}_{jlm} HB^h{}_{itk} \\ (4.4) \qquad & - HB^t{}_{klm} HB^h{}_{ijt} + D^h{}_{tlm} HB^t{}_{ijk} - D^t{}_{ilm} HB^h{}_{tjk} \\ & - D^t{}_{ilm} HB^h{}_{itk} + D^t{}_{klm} HB^h{}_{ijt} = 0 \end{aligned}$$

Differentiating (4.7) covarianty and using (3.14) and the fact that the HB-tensor is parallel, we get

$$\chi_s(HB^t{}_{ijk} D^h{}_{tlm} - HB^h{}_{tjk} D^t{}_{ilm} - HB^h{}_{itk} D^t{}_{jlm} - HB^h{}_{ijt} D^t{}_{klm}) = 0$$

and as $\chi_s \neq 0$ for at least one s, then

$$(4.5) \qquad HB^t{}_{ijk} D^h{}_{tlm} - HB^h{}_{tjk} D^t{}_{ilm} - HB^h{}_{itk} D^t{}_{jlm} - HB^h{}_{ijt} D^t{}_{klm} = 0$$

Contracting in indices h and m, we get:

$$HB^t{}_{ijk} D^m{}_{tlm} = (n+4)\mathrm{II}_{lt} HB^t{}_{ijk} + \mathrm{II} HB_{lijk}$$

$$\begin{aligned} HB^h{}_{tjk} D^t{}_{ilh} = & -HB^h{}_{ljk}\mathrm{II}_{hi} + g_{li}\mathrm{II}_h{}^t HB^h{}_{tjk} - HB_{itjk}\mathrm{II}_l{}^t + HB^h{}_{tjk} F_h{}^t T_{li} \\ & - HB^h{}_{tjk} F_l{}^t T_{hi} + HB^h{}_{tjk} F_{li} T_h{}^t - HB^h{}_{tjk} F_{hi} T_l{}^t \\ & + 2HB^h{}_{tjk} T_i{}^t F_{lh} + T_{lh} F_i{}^t HB^h{}_{tjk} \end{aligned}$$

$$\begin{aligned} HB^h{}_{itk} D^t{}_{jlh} = & -HB^h{}_{ilk}\mathrm{II}_{hj} + g_{li} HB^h{}_{itk}\mathrm{II}_h{}^t - \mathrm{II}_l{}^t HB_{jitk} + F_h{}^t T_{lj} HB^h{}_{itk} \\ & - HB^h{}_{itk} F_l{}^t T_{hj} + HB^h{}_{itk} F_{lj} T_h{}^t - HB^h{}_{itk} F_{hj} T_l{}^t \\ & + 2HB^h{}_{itk} T_j{}^t F_{lh} + T_{lh} F_j{}^t HB^h{}_{itk} \end{aligned}$$

$$\begin{aligned} HB^h{}_{ijt} D^t{}_{klj} = & -HB^h{}_{ijl}\mathrm{II}_{hk} + g_{lk}\mathrm{II}_h{}^t HB^h{}_{ijt} - HB_{kijt}\mathrm{II}_l{}^t + HB^h{}_{ijt} F_h{}^t T_{lk} \\ & - F_l{}^t T_{hk} HB^h{}_{ijt} + F_{lk} T_h{}^t HB^h{}_{ijt} - F_{hk} T_l{}^t HB^h{}_{ijt} \\ & + 2T_k{}^t F_{lh} HB^h{}_{ijt} + T_{lh} F_k{}^t HB^h{}_{ijt} \end{aligned}$$

According to all these equalities, (4.8) can be rewritten in this way:

$$
\begin{aligned}
&(n+4)\mathrm{II}_{lt}HB^{t}{}_{ijk} + \mathrm{II}HB_{lijk}\\
&+\mathrm{II}_l{}^{t}(HB_{itjk} + HB_{ijkt} + HB_{iktj})\\
&+HB^{h}{}_{ljk}\mathrm{II}_{hi} + HB^{h}{}_{ilk}\mathrm{II}_{hj}HB^{h}{}_{ijl}\mathrm{II}_{hk}\\
&-F_h{}^{t}(HB^{h}{}_{tjk}T_{li} + HB^{h}{}_{itk}T_{lj}HB^{h}{}_{ijt}T_{lk})\\
&-F_l{}^{t}(HB^{h}{}_{tjk}T_{hi} + HB^{h}{}_{itk}T_{hj}HB^{h}{}_{ijt}T_{hk})\\
&-T_l{}^{t}(HB^{h}{}_{tjk}F_{hi} + HB^{h}{}_{itk}F_{hj}HB^{h}{}_{ijt}F_{hk})\\
&-2F_{lh}(HB^{h}{}_{tjk}T_i{}^{t} + HB^{h}{}_{itk}T_j{}^{t} + HB^{h}{}_{ijt}T_k{}^{t})\\
&-2T_{lh}(HB^{h}{}_{tjk}F_i{}^{t} + HB^{h}{}_{itk}F_j{}^{t} + HB^{h}{}_{ijt}F_k{}^{t}) = 0
\end{aligned}
$$

Using the Bianchi identity for components of HB-tensor, other algebraic properties and relations (4.1), (4.2) and (4.3), we can get from the upper equality

$$(n+4)\mathrm{II}_{lt}HB^{t}{}_{ijk} + \mathrm{II}HB_{lijk} = 0 \tag{4.6}$$

Taking into account the recurrence of the tensor II_{ij} (Lemma 1.1), we can obtain:

$$\chi^{a}\mathrm{II}_{aj} = \frac{n+1}{n+4}\mathrm{II}\chi_j \tag{4.7}$$

Transvecting (4.9) by χ^{l}, we obtain

$$(n+2)\mathrm{II}\chi_l HB^{l}{}_{ijk} = 0 \tag{4.8}$$

and, from this, yields

$$\mathrm{II} = 0 \text{ (what is equivalent to } K = 0) \tag{4.9}$$

or

$$\chi_l HB^{l}{}_{ijk} = 0 \tag{4.10}$$

Suppose that the relation (4.13) is fulfilled.

As the space is HB parallel, then

$$HB^{a}{}_{jkl,a} = 0 \tag{4.11}$$

Using (0.4) we obtain

$$(4.12)\qquad \begin{aligned} &(n+3)(\chi_l \mathrm{II}_{kj} - \chi_k \mathrm{II}_{jl}) + \frac{3\mathrm{II}}{n+4}(\chi_l g_{kj} - \chi_k g_{jl}) \\ &+\sigma_k T_{lj} - \sigma_l T_{kj} - 2\sigma_j T_{kl} - \frac{n+1}{n+4}(\sigma_k F_{lj} + \sigma_l F_{kj} - 2\sigma_j F_{kl}) = 0 \end{aligned}$$

where σ_k stands for the vector

$$(4.13)\qquad \sigma_k = \chi_a F_j{}^a$$

We are going to prove the following

Lemma 4.2. *If* $\chi_l HB^l{}_{ijk} = 0$ *then* $\sigma_l HB^l{}_{ijk} = 0$.

Proof. As (4.16) and (4.3) hold, then

$$\begin{aligned} \sigma_l HB^l{}_{ijk} &= \chi_a F_l{}^a HB^l{}_{ijk} = \chi^a F_{la} HB^l{}_{ijk} \\ &= \chi^a F_{li} HB^l{}_{ajk} = \chi^a F^l{}_i HB_{lajk} = \chi^a F_i{}^l HB^l{}_{aljk} = 0 \qquad \blacksquare \end{aligned}$$

Now we shall transvect (4.15) by $HB^l{}_{mtp}$. We obtain

$$\begin{aligned} &-(n+3)\chi_k \mathrm{II}_{jl} HB^l{}_{mtp} - \frac{3\mathrm{II}}{n+4}\chi_k HB_{jmtp} + \sigma_k T_{lj} HB^l{}_{mtp} - 2\sigma_j T_{kl} HB^l{}_{mtp} \\ &-\frac{n+1}{n+4}(\sigma_k F_{lj} HB^l{}_{mtp} - 2\sigma_j F_{kl} HB^l{}_{mtp}) = 0 \end{aligned}$$

Taking into account (4.9) and (3.4), we can rewrite this relation in the following form:

$$\frac{\mathrm{II}}{n+4}[-(n+6)\chi_k HB_{jmtp} + n\sigma_k HB_{amtp} + 2(n+2)\sigma_j HB_{amtp} F_k{}^a] = 0$$

As (4.11) holds and we have supposed that there holds (4.13) and (4.12), then our relation gives:

$$(4.14)\qquad (n+6)\chi_k HB_{jmtp} = n\sigma_k HB_{amtp} + 2(n+2)\sigma_j HB_{amtp} F_k{}^a$$

n the hyperbolic Kaehlerian space, which we consider, there exists a vector field (λ_k)such that $\lambda_k \chi^k = 1$. we shall prove the following

Lemma 4.3. *The vector field* (λ_k) *satisfying the relation*

$$(4.15)\qquad \lambda_k \chi^k = 1$$

is orthogonal to the vector (σ_k), *defined by (4.16).*

Proof. $\sigma_k = F_k{}^a \chi_a \lambda^k = F^{ka}\chi_a\lambda_k = -F^{ak}\chi_a\lambda_k$ ■

Now we shall transvect the relation (4.17) by λ^k. We obtain

$$(n+6)HB_{jmtp} = 2(n+2)\sigma_j\mu^a HB_{amtp} \tag{4.16}$$

where

$$\mu^a = \lambda^k F_k{}^a \tag{4.17}$$

In the same time, the tensor HB_{jmtp} is skew symmetric in the first two indices and then there yields from (4.19)

$$\sigma_j\mu^a HB_{amtp} + \sigma_m\mu^a HB_{ajtp} = 0 \tag{4.18}$$

From the relation (4.19), transvecting by λ^k, according to Lemma 3.3 we obtain

$$\lambda^k HB_{jmtp} = 0 \tag{4.19}$$

Now we will transvect (4.17) by λ^j. We obtain, because of (4.22) and Lemma 4.3

$$\sigma_k\mu^a HB_{amtp} = 0 \tag{4.20}$$

and, according to (4.19),

$$HB_{kmtp} = 0 \tag{4.21}$$

Then, if in a HB-parallel Ricci-recurrent hyperbolic Kaehlerian space the Ricci tensor does not vanish, then the scalar curvature of such a space vanishes or the HB-tensor vanishes. If the HB-tensor vanishes, than such a space is the recurrent one, by the fact of (3.5).

$$K_{kmtp} = D_{kmtp} \tag{4.22}$$

and this one is recurrent.

If $\mathrm{II} = 0$ then $K = 0$. In this, the last one, case we can get from (4.15)

$$\chi_l \mathrm{II}_{kj} - \chi_k \mathrm{II}_{jl} = \frac{1}{n+3}(\sigma_k T_{jl} + \sigma_l T_{kj} + 2\sigma_j T_{kl}) \tag{4.23}$$

The vectors (χ_l) and (σ_l) are mutually orthogonal ([6]). There holds

$$\mathrm{II}_{jl}\chi^l = \frac{n+1}{2(n+2)(n+4)}K\chi_j \; ; \quad T_{jl}\chi^l = \frac{n+1}{2(n+2)(n+4)}K\sigma_j \tag{4.24}$$

Transvecting the relation (4.26) by χ^l, we obtain

$$\chi_l \chi^l \mathrm{II}_{kj} = 0 \tag{4.25}$$

Suppose that $\mathrm{II}_{kj} = 0$. Then

$$K = \frac{K}{2(n+2)} g_{kj}$$

and after transvection by g^{kj},

$$K = \frac{nK}{2(n+2)}$$

From these equalities we can see that $K = 0$ and $K_{kj} = 0$. As we considered that case separately in the Theorem 4.1, we are not interested in it now. Then, from(4.28) we have

$$\chi_l \overset{*}{\chi}{}^l = 0 \tag{4.26}$$

As a survey of all this considerations, we can say shortly:

If a HB-parallel hyperbolic Kaehlerian space is non trivially Ricci-recurrent with the recurrence vector χ_l, then the following cases may occur:

(1) The space is HB-flat and recurrent

(2) The space is symmetric in the sense of Cartan and $K_{ij} = 0$

(3) The scalar curvature of the space vanishes and the vector (χ_l) is an isotropic vector field.

§5. Conclusions about Ricci-recurrent HB-parallel hyperbolic Kaehlerian spaces

Our classification from previous paragraph is fully analogous to the classification theorem from [1]. The last case (3), is an essential case. Using the classification theorem from [1], W. Roter proved the existence of an essential conformally symmetric Ricci-recurrent pseudo-Riemannian space, by constructing its metric tensor ([7]). But the analogy between conformally conformally symmetric Ricci-recurrent pseudo-Riemannian spaces and HB-parallel Ricci-recurrent hyperbolic Kaehlerian spaces is not so large, for the sake of results in §1.

If we apply the result of the Theorem 1.1 to the case (3) ("essential") from the classification, then we can conclude that the essential case does not exist, because the scalar curvature vanishes and gives us a symmetric space.

Further, we are going to make some more considerations of the case (1) of the classification. If the hyperbolic Kaehlerian space is HB-flat, then its curvature tensor can be expressed in this way:

$$
\begin{aligned}
K^i{}_{jkl} = &\frac{1}{n+4}[\delta_l{}^i K_{kj} - \delta_k{}^i K_{lj} + g_{kj}K_l{}^i - g_{lj}K_k{}^i + F_l{}^i S_{kj} + F_k{}^i S_{lj} \\
&F_{kj}S_l{}^i + F_{lj}S_k{}^i + 2S_j{}^i F_{kl} + 2S_{kl}F_j{}^i \\
&- \frac{K}{n+2}(\delta_l{}^i g_{kj} - \delta_k{}^i g_{lj} + F_{lj}F_k{}^i - F_l{}^i F_{kj} - 2F_j{}^i F_{kl})
\end{aligned}
\tag{5.1}
$$

As we supposed the HB-parallel hyperbolic Kaehlerian space is Ricci-recurrent, Then it is recurrent, according to (5.1), (0.1) and (0.5). As the recurrence vector (χ_k) is a gradient, then there holds

$$
K_{mi}K^i{}_{skl} + K_{si}K^i{}_{mkl} = 0 \text{ (Ricci identity)} \tag{5.2}
$$

We obtain from (5.1), (5.2)

$$
\begin{aligned}
&g_{ks}K_l{}^i K_{mi} - g_{ls}K_k{}^i K_{mi} + F_{ks}S_l{}^i K_{mi} - F_{ls}S_k{}^i K_{mi} + 2S_s{}^i K_{mi}F_{kl} \\
&-\frac{K}{n+2}(K_{ml}gks - K_{mk}gks + F_{ls}Smk - F_{ks}Sml) \\
&+g_{km}K_l{}^i K_{si} - g_{lm}K_l{}^i K_{si} + F_{km}S_l{}^i K_{si} - F_{lm}S_k{}^i K_{si} + 2S_m{}^i K_{si}F_{kl} \\
&-\frac{K}{n+2}(K_{sl}gkm - K_{sk}gkm + F_{lm}Ssk - F_{km}Ssl) = 0
\end{aligned}
$$

If the space is Ricci recurrent and if the recurrence vector is a gradient, then there holds ([1], [7])

$$
K_l{}^i K_{mi} = 1/2\ KK_{lm} \tag{5.3}
$$

and, consequently,

$$
S_l{}^i K_{mi} = 1/2\ KS_{lm} \tag{5.4}
$$

Even we substitute (5.3) and (5.4) into the upper expression, then there holds

$$
\begin{aligned}
&K(g_{ks}K_{lm} - g_{ls}K_{km} + g_{km}K_{ls} - g_{lm}K_{ks} \\
&+F_{ks}S_{lm} - F_{ls}S_{km} + F_{km}S_{ls} - F_{lm}S_{ks}) = 0
\end{aligned}
$$

From this expression we have

$$K = 0 \tag{5.5}$$

or

$$\begin{aligned} & g_{ks}K_{lm} - g_{ls}K_{km} + g_{km}K_{ls} - g_{lm}K_{ks} \\ & +F_{ks}S_{lm} - F_{ls}S_{km} + F_{km}S_{ls} - F_{lm}S_{ks} = 0 \end{aligned} \tag{5.6}$$

If (5.5) is fulfilled the space will be symmetric according to Theorem 1.1. As it is a recurrent space in the same time, then its recurrence vector (χ_k) will vanish.

Suppose, now, that (5.6) is satisfied. We shall transvect this equality by $F_r{}^k$. We obtain

$$\begin{aligned} & F_{rs}K_{lm} - g_{ls}S_{mr} + F_{rm}K_{ls} - g_{lm}S_{sr} \\ & +g_{sr}S_{lm} + F_{ls}K_{mr} + g_{mr}S_{ls} + F_{lm}K_{sr} = 0 \end{aligned}$$

After transvection by $F_t{}^m$, we obtain

$$\begin{aligned} & F_{rs}S_{lt} + g_{ls}K_{rt} - g_{rt}K_{ls} - F_{tl}S_{sr} \\ & +g_{sr}K_{lt} + F_{ls}S_{rt} + F_{tr}S_{ls} - g_{lt}K_{sr} = 0 \end{aligned}$$

In this equality, we substitute the index r for k and t for m. Then there holds

$$\begin{aligned} & F_{ks}S_{lm} + g_{ls}K_{km} - g_{km}K_{ls} - F_{ml}S_{sk} \\ & +g_{sk}K_{lm} + F_{ls}S_{km} + F_{mk}S_{ls} - g_{lm}K_{sk} = 0 \end{aligned} \tag{5.7}$$

Subtracting (5.7) of (5.6), we obtain

$$2(F_{km}S_{ls} - F_{ls}S_{km} - g_{ls}K_{km}) = 0$$

what means

$$K_{km}g_{ls} = F_{km}S_{ls} - F_{ls}S_{km}) \tag{5.8}$$

If we transvect (5.8) by g^{ls}, then

$$nK_{km} = 0$$

i.e.

$$K_{km} = 0 \tag{5.9}$$

From (5.9) and (5.1) we have

$$K^{i}{}_{jkl} = 0 \tag{5.10}$$

i.e. the space is flat.

Now we can give more proper form of the classification from §4:

Theorem 5.1. *If a hyperbolic Kaehlerian space is a HB-parallel and Ricci recurrent, then its scalar curvature vanishes. In that case, one of these possibilities hold:*

(1) The space is HB-flat Ricci-flat and flat

(2) The space is symmetric in the sense of Cartan and Ricci flat

Bibliography

[1] T. Adati, T. Miyzawa. On conformally symmetric spaces. *Tensor, N. S.* **18**, 1967, 335–342.

[2] N. Barros, A. Romero. Indefinite Kaehlerian manifolds. *Math. Ann.* **261**, 1982, 55-62

[3] N. C. L. Bejan (in Romanian). Structuri Hiperbolice pe diverse spatii fibrate, resumatul tezei de doctorat, Iasi 1990.

[4] C. L. Bejan . Almost parahermitian structure on the tangent bundle of an almost paracohermitian manifold. Proceedings of the fifth national seminar of Finsler and Lagrange spaces, Brasov 1988, 105-109

[5] B. Chen, K. Yano. On Kaehlerian spaces with parallel or vanishing scalar curvature. *J. Math. Soc. Japan* **27**, 1975, 106-112

[6] G. Gjelepov (in Bulgarian) Холоморфни проективности, конформности и рекурентности в псевдориманови пространства с допълнителна структура, Пловдив 1980 (master thesis).

[7] G. Ganchev, A. Borisov. Isotropic sections and curvative properties of hyperbolic Kaehlerian manifolds, *Publ,de l' Inst. Math.*, **38 (52)**, 1985, 183-192

[8] Y. Kubo. Kaehlerian manifolds with vanishing Bochner curvature tensor *Kodai Math. Sem. Rep.*, **28**, 1976, 85-89

[9] G. Markov, G. Gjelepov (in Bulgarian) Холоморфни проективни, преобразования на Келерови пространства в симетрични Келерови пространства. Пловдивски университет "Паисий Хилендарски", **16**, 1978, 443-453

[10] M. Matsumoto. On Kaehlerian spaces with parallel or vanishing Bochner curvature tensor. *Tensor. N. S.*, **20**, 1969, 25-28

[11] M. Matsumoto., S. Tanno. Kaehlerian spaces with parallel or vanishing Bochner curvature tensor *Tensor. N. S.*, **20**, 1973, 291-294

[12] E. Pavlov., (in Russian). Вещественая реализация конформного соответствия Римановых пространств над Клиффордовой алгеброй. *Известия ВУЗ*, **7(194)**, 1978, 64-67.

[13] N. Pušić. On invariant tensor of a conformal transformation of a hyperbolic Kaehlerian manifold. *Zbornik radova Filozofkog fakulteta w Nišu serija Matematika* **4**, 1990, 55-64.

[14] R. K. Rashevski (in Russian). Скалярое поле в расслоенном пространстве. *Труды семинара по векторному и тензорному анализу*, **6**, 1948, 225-248.

[15] W. Roter. On conformally symmetric Ricci recurent spaces, *Colloquium Math.*, vol. **XXXI**, fasc. 1, 1974, 87-96

[16] H. S. Ruse, G. Walker. T. S. Willmore. Harmonic spaces. Edizioni Cremonese, Roma, 1961.

[17] A. P. Shirokov (in Russian). Геометрия обобщених биаксиальных пространств над. *Учл записки Казанского г. у.*, **114**, кн. 2, 1954, 123-165.

[18] K. Y a n o. On complex conformal connections, *Kodaj Math. Sem. Rep.*, **26**, 1975, 137-151
[19] K. Y a n o. Differential geometry of complex and almost complex spaces. Pergamon Press, New York 1965.

Matematički institut
Prir. matem. fakultet
University of Novi Sad
Novi Sad
YUGOSLAVIA

Received 02.07.1992