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The Limit Case of Bernstein's Operators with Jacobi–weights

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1. Introduction

The Bernstein - type operators discussed in this paper are given by

$$U_n f \equiv U_n(f,x) = f(0)P_{n,o}(x) + f(1)P_{n,n}(x) + \sum_{k=1}^{n-1} P_{n,k}(x) \int_0^1 (n-1)P_{n-2,k-1}(t)f(t)\,dt,$$

where

$$P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

The quantity $(n-1)\int_0^1 P_{n-2,k-1}(t)f(t)\,dt$ for $1 \le k \le n-1$ in the operators $U_n(f,x)$ takes place of $f(\frac{k}{n})$ in $B_n(f,x)$, the Bernstein polinomials. These operators were introduced by T.N.T. Goodman and A. Sharma [3]. In this paper we shall study the relation between the rate of approximation of $U_n(f,x)$ and the K-functional

$$K\Big(f,\frac{1}{n}\Big)_\infty \equiv K\Big(f,\frac{1}{n};L_\infty,W_\infty^2(\varphi)\Big) = \inf\Big\{\|f-g\|_\infty + \frac{1}{n}\|\varphi g''\|_\infty; g \in W_\infty^2(\varphi)\Big\},$$

where

$$W_\infty^2(\varphi) = \Big\{f \in AC[0,1]; f' \in AC_{loc}[0,1], \varphi f'' \in L_\infty[0,1]\Big\}$$

with $\varphi(x) = x(1-x)$. We shall prove a direct inequality and strong converse inequality of type A. Here the terminology of [2] is used. So, we shall show

$$\|U_n f - f\|_\infty \sim K\Big(f, \frac{1}{n}\Big)_\infty, \tag{1.1}$$

i.e. we shall find two constants c_1 and c_2 such that

$$c_1\|f - U_n f\|_\infty \le K\Big(f, \frac{1}{n}\Big)_\infty \le c_2\|f - U_n f\|_\infty. \tag{1.2}$$

2. Some basic properties

We recall some properties proved in [3,4].

(2.1) U_n is linear and positive.

(2.2) $U_n(1,x) = 1\ ,\ \ U_n(t,x) = x.$

(2.3) $U_n(t^2,x^2) = x^2 + \dfrac{2\varphi(x)}{n+1}.$

(2.4) $\|U_n f\|_\infty \le \|f\|_\infty.$

(2.5) $f \le U_k f \le U_n f$ for convex f and natural $k \ge n$.

(2.6) $U_k U_n f = U_n U_k f$, i.e. U_n commutes with U_k.

3. The Jackson-type inequality and the direct theorem

In order to show the direct part of $\|U_n f - f\|_\infty \sim K\big(f, \frac{1}{n}\big)_\infty$, i.e. the first inequality in (1.2), we prove the following three lemmas concerning any operator L satisfying condition (3.1).

(3.1) (1) L is linear and positive operator from $L_\infty[0,1]$ to $L_\infty[0,1]$,

(2) $L(1,x) = 1,\ \ L(t,x) = x.$

From (1) and (2) we obtain the following property

(3) $Lf \ge f$ for convex f.

Lemma 3.1. *Let*

$$K_y(x) = \begin{cases} y(x-1) & \text{for } 0 \le y \le x \le 1 \\ x(y-1) & \text{for } 0 \le x \le y \le 1 \end{cases}$$

If $f \in W^2_\infty(\varphi)$, *then we have*

$$L(f,x) - f(x) = \int_0^1 (L(K_y,x) - K_y(x))f''(y)\,dy. \tag{3.2}$$

Proof. We use the idea of the proof of Lemma 1 in [5]. From condition (3.1) we conclude that L is a bounded operator and we obtain

$$\int_0^1 L\Big(K_y(t,x)\Big)f''(y)\,dy = L\Big(\int_0^1 K_y(t,x)f''(y)\,dy\Big).$$

Now we have

$$\begin{aligned}\int_0^1 K_y(x)f''(y)\,dy &= \int_0^x y(x-1)f''(y)\,dy + \int_x^1 x(y-1)f''(y)\,dy \\ &= f(x) - (1-x)f(0) - xf(1).\end{aligned} \tag{3.3}$$

$$\begin{aligned}\int_0^1 L(K_y(t,x))f''(y)\,dy &= L(f,x) - f(0)L((1-t),x) - f(1)L(t,x) \\ &= L(f,x) - f(0)(1-x) - f(1)x.\end{aligned} \tag{3.4}$$

The equations (3.3) and (3.4) yield (3.2). Lemma 3.1 is proved.

Lemma 3.2. *Let* $f_0(x) = x\ln x + (1-x)\ln(1-x)$. *Then*

$$\|Lf - f\|_\infty \le \|f''\varphi\|_\infty \|Lf_0 - f_0\|_\infty$$

for every $f \in W^2_\infty(\varphi)$.

Proof. $K_y(x)$ is convex and nonpositive then from (3.1) it follows that $L(K_y(\cdot),x) - K_y(x) \ge 0$. From Lemma 3.1 we have

$$L(f,x) - f(x) = \int_0^1 \frac{L(K_y(t),x) - K_y(x)}{\varphi(y)} f''(y)\varphi(y)\,dy.$$

Thus

$$\|Lf - f\|_\infty \le \|f''\varphi\|_\infty \max_{x\in[0,1]}\Big|L\Big(\int_0^1 \frac{K_y(t)}{\varphi(y)}\,dy, x\Big) - \int_0^1 \frac{K_y(x)}{\varphi(y)}\,dy\Big| \tag{3.5}$$

and

$$\int_0^1 \frac{K_y(x)}{\varphi(y)}\,dy = \int_0^x \frac{y(x-1)}{y(1-y)}\,dy + \int_x^1 \frac{x(y-1)}{y(1-y)}\,dy = f_0(x). \tag{3.6}$$

As a corollary of (3.5) and (3.6) we have

$$\|Lf - f\|_\infty \le \|f''\varphi\|_\infty \max_{x\in[0,1]} |L(f_0, x) - f_0(x)| = \|Lf_0 - f_0\|_\infty \|f''\varphi\|_\infty.$$

Lemma 3.3. *Let f_0 be the function from Lemma 3.2. Then the following inequality holds true*

$$\|Lf_0 - f_0\|_\infty \le \left\|\frac{1}{\varphi(\cdot)} L((t-\cdot)^2, \cdot)\right\|_\infty.$$

Proof.

$$\begin{aligned} L(f_0(t), x) - f_0(x) &= L\Big((1-t)\ln(1-t) + t\ln t, x\Big) \\ &\quad - L(1-t, x)\ln(1-x) - L(t, x)\ln x \\ &= L\Big((1-t)\ln(1-t) - (1-t)\ln(1-x), x\Big) \\ &\quad + L(t\ln t - t\ln x, x). \end{aligned} \tag{3.7}$$

Expanding $\ln(x + t - x)$ by Taylor's formula

$$\ln t = \ln x + \frac{(t-x)}{x} - \int_x^t \frac{(t-u)}{u^2}\, du$$

and using $\int_x^t \frac{(t-u)}{u^2}\, du \ge 0$ we have:

$$t\ln t - t\ln x \le \frac{t(t-x)}{x} \tag{3.8}$$

$$(1-t)\ln(1-t) - (1-t)\ln(1-x) \le \frac{(1-t)(x-t)}{(1-x)} \tag{3.9}$$

After substitution of (3.8) and (3.9) in (3.7) we get

$$0 \le L(f_0(t), x) - f_0(x) \le L\Big(\frac{(1-t)(x-t)}{(1-x)} + \frac{t(t-x)}{x}, x\Big)$$

and therefore

$$0 \le L(f_0(t), x) - f_0(x) \le \frac{1}{\varphi(x)} L((t-x)^2, x).$$

Hence

$$\|Lf_0 - f_0\|_\infty \le \left\|\frac{1}{\varphi(\cdot)} L((t-\cdot)^2, \cdot)\right\|_\infty.$$

As a direct consequence of Lemma 3.2 and Lemma 3.3 we obtain the following theorem

Theorem 3.1. (Jackson-type inequality). *For every $f \in W^2_\infty(\varphi)$ we have*

$$\|Lf - f\|_\infty \leq \|f''\varphi\|_\infty \Big\| \frac{1}{\varphi(.)} L((t-.)^2,.)\Big\|_\infty.$$

Let us put $L = U_n$ in Theorem 3.1. Then

$$\frac{1}{\varphi(x)} U_n((t-x)^2,x) = \frac{U_n(t^2,x) - x^2}{\varphi(x)} = \frac{2}{(n+1)}$$

and therefore

$$\|U_n f - f\|_\infty \leq \frac{2}{(n+1)} \|f''\varphi\|_\infty. \tag{3.10}$$

Theorem 3.2. (A direct-type theorem). *For every $f \in L_\infty[0,1]$ we have*

$$\|U_n f - f\|_\infty \leq 2K\Big(f, \frac{1}{n}\Big)_\infty. \tag{3.11}$$

P r o o f. Let $g \in W^2_\infty(\varphi)$. Then

$$\|U_n f - f\|_\infty \leq \|U_n f - U_n g\|_\infty + \|U_n g - g\|_\infty + \|g - f\|_\infty.$$

Using (2.4) and (3.10) we get

$$\|U_n f - f\|_\infty \leq 2\|f-g\|_\infty + \frac{2}{(n+1)}\|g''\varphi\|_\infty \leq 2\Big(\|f-g\|_\infty + \frac{1}{n}\|g''\varphi\|_\infty\Big). \tag{3.12}$$

Taking an infimum on all $g \in W^2_\infty(\varphi)$ in (3.12) we derive (3.11).

4. Two important properties of $U_n f$

Lemma 4.1. *For every $f \in L_\infty[0,1]$ we have*

$$U_{n-1}(f,x) - U_n(f,x) = \frac{\varphi(x)}{n(n-1)} U''_n(f,x).$$

P r o o f. Let

$$U_n(f,x) = A_n(f,x) + D_n(f,x),$$

where

$$A_n(f,x) = (1-x)^n f(0) + x^n f(1)$$

and

$$D_n(f,x) = \sum_{k=1}^{n-1} P_{n,k}(x) \int_0^1 (n-1)P_{n-2,k-1}(t)f(t)\,dt.$$

Then

$$\begin{aligned}
D_{n-1}(f,x) &= \sum_{k=1}^{n-2} P_{n-1,k}(x) \int_0^1 (n-2)P_{n-3,k-1}(t)f(t)\,dt \\
&= \sum_{k=1}^{n-2} P_{n-1,k}(x) \int_0^1 (n-2)\binom{n-3}{k-1} t^k(1-t)^{n-k-2} f(t)\,dt \\
&+ \sum_{k=1}^{n-2} P_{n-1,k}(x) \int_0^1 (n-2)\binom{n-3}{k-1} t^{k-1}(1-t)^{n-k-1} f(t)\,dt \\
&= \sum_{k=2}^{n-1} P_{n-1,k-1}(x) \int_0^1 (n-2)\binom{n-3}{k-2} t^{k-1}(1-t)^{n-k-1} f(t)\,dt \\
&+ \sum_{k=1}^{n-2} P_{n-1,k}(x) \int_0^1 (n-2)\binom{n-3}{k-1} t^{k-1}(1-t)^{n-k-1} f(t)\,dt \\
&= \sum_{k=1}^{n-1} P_{n,k}(x) \int_0^1 (n-1)P_{n-2,k-1}(t)f(t)\,dt \\
&\times \left(\frac{1}{x}\frac{k(k-1)}{n(n-1)} + \frac{1}{(1-x)}\frac{(n-k)(n-k-1)}{n(n-1)}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
D_{n-1}(f,x) - D_n(f,x) &= \sum_{k=1}^{n-1} P_{n,k}(x) \int_0^1 (n-1)P_{n-2,k-1}(t)f(t)\,dt \\
&\times \frac{(1-x)k(k-1) + x(n-k)(n-k-1) - n(n-1)\varphi(x)}{\varphi(x)n(n-1)} \\
&= \sum_{k=1}^{n-1} P_{n,k}(x) \int_0^1 (n-1)P_{n-2,k-1}(t)f(t)\,dt \\
&\times \left(\frac{(k-nx)^2}{\varphi^2(x)} - \frac{n\varphi(x) + (k-nx)(1-2x)}{\varphi^2(x)}\right)\frac{\varphi(x)}{n(n-1)}.
\end{aligned}$$

On the other hand

$$P'_{n,k}(x) = \frac{k-nx}{\varphi(x)} P_{n,k}(x)$$

and

$$P''_{n,k}(x) = P_{n,k}(x)\left(\frac{(k-nx)^2}{\varphi^2(x)} - \frac{n\varphi(x)+(k-nx)(1-2x)}{\varphi^2(x)}\right) \tag{4.1}$$

and therefore

$$D_{n-1}(f,x) - D_n(f,x) = \frac{\varphi(x)}{n(n-1)} D''_n(f,x) \tag{4.2}$$

for every $f \in L_\infty[0,1]$. For the other part of $U_{n-1}(f,x) - U_n(f,x)$ we have
(4.3)

$$A_{n-1}(f,x) - A_n(f,x) = f(0)x^{n-1}(1-x) + f(1)x(1-x)^{n-1} = \frac{\varphi(x)}{n(n-1)} A''_n(f,x)$$

for every $f \in L_\infty[0,1]$. Now using (4.2) and (4.3) we prove Lemma 4.1.

Remark. Lemma 4.1 is a limit case of Theorem 5 in [1]. The method used in the proof allows us to put less conditions on the function f than in [1].

Let

$$W^2_\infty(\varphi)\{0;1\} = \{f \in W^2_\infty(\varphi) \ : \ \lim_{x\downarrow 0}(\varphi(x)f''(x)) = 0 \ , \ \lim_{x\uparrow 1}(\varphi(x)f''(x)) = 0\}.$$

Lemma 4.2. *For every $f \in W^2_\infty(\varphi)\{0;1\}$ we have*

$$\varphi(x)U''_n(f,x) = U_n(\varphi f'', x),$$

i.e. U_n commutes with the operator D, given by $Df := \varphi f''$.

Proof. We have $U_n f = \sum_{k=0}^n u_k^{(n)}(f)P_{n,k}$, where $u_0^{(n)}(f) = f(0)$, $u_n^{(n)}(f) = f(1)$ and $u_k^{(n)}(f) = \int_0^1 (n-1)P_{n-2,k-1}(t)f(t)\,dt$ if $1 \le k \le n-1$.
Let $\Delta^1 u_k^{(n)}(f) = u_{k+1}^{(n)}(f) - u_k^{(n)}(f)$ and $\Delta^2 u_k^{(n)}(f) = \Delta^1(\Delta^1 u_k^{(n)}(f))$, there-

fore

$$\begin{aligned}
U_n'(f,x) &= \sum_{k=0}^{n} u_k^{(n)}(f)\binom{n}{k}\left(kx^{k-1}(1-x)^{n-k}-(n-k)x^k(1-x)^{n-k-1}\right)\\
&= \sum_{k=1}^{n}\binom{n}{k}u_k^{(n)}(f)kx^{k-1}(1-x)^{n-k}-\sum_{k=0}^{n-1}\binom{n}{k}u_k^{(n)}(f)(n-k)x^k(1-x)^{n-k-1}\\
&= n\sum_{k=1}^{n}\binom{n-1}{k-1}u_k^{(n)}(f)x^{k-1}(1-x)^{n-k}-n\sum_{k=0}^{n-1}\binom{n-1}{k}u_k^{(n)}(f)x^k(1-x)^{n-k-1}\\
&= n\sum_{k=0}^{n-1}(u_{k+1}^{(n)}(f)-u_k^{(n)}(f))P_{n-1,k}(x)\\
&= n\sum_{k=0}^{n}(u_{k+1}^{(n)}(f)-u_k^{(n)}(f))P_{n-1,k}(x)\\
&= n\sum_{k=0}^{n-1}\Delta^1 u_k^{(n)}(f)P_{n-1,k}(x).
\end{aligned}$$

From the above representation we have

$$U_n''(f,x) = n(n-1)\sum_{k=0}^{n-2}\Delta^2 u_k^{(n)}(f)P_{n-2,k}(x). \tag{4.4}$$

Now we shall prove that

$$\Delta^2 u_k^{(n)}(f) = \frac{1}{n}\int_0^1 P_{n,k+1}(t)f''(t)\,dt \tag{4.5}$$

for $0 \le k \le n-2$. In the case $1 \le k \le n-3$ using integration by parts we obtain

$$\begin{aligned}
\Delta^2 u_k^{(n)}(f) &= \int_0^1 (n-1)(P_{n-2,k+1}(t)-2P_{n-2,k}(t)+P_{n-2,k-1}(t))f(t)\,dt\\
&= \frac{1}{n}\int_0^1 P_{n,k+1}''(t)f(t)\,dt = \frac{1}{n}\int_0^1 P_{n,k+1}(t)f''(t)\,dt\,.
\end{aligned}$$

In cases k=0 and k=n-2 we have the additional terms f(0) and f(1), respectively. But they are canceled out with $-\frac{f(0)P_{n,1}'(0)}{n}$ and $\frac{f(1)P_{n,n-1}'(1)}{n}$. Thus, equality

(4.5) holds true for $0 \le k \le n-2$. From (4.4) and (4.5) we get

$$U_n''(f,x) = \sum_{k=0}^{n-2} P_{n-2,k}(x) \int_0^1 (n-1)P_{n,k+1}(t)f''(t)\,dt$$

and therefore

$$\begin{aligned}\varphi(x)U_n''(f,x) &= \sum_{k=0}^{n-2} P_{n,k+1}(x) \int_0^1 (n-1)P_{n-2,k}(t)\varphi(t)f''(t)\,dt \\ &= \sum_{k=1}^{n-1} P_{n,k}(x) \int_0^1 (n-1)P_{n-2,k-1}(t)\varphi(t)f''(t)\,dt \\ &= U_n(\varphi f'', x),\end{aligned}$$

because $f \in W_\infty^2(\varphi)\{0;1\}$. Lemma 4.2 is proved.

Using (3.10) for $f \in W_\infty^2(\varphi)$ we have $\lim_{n\to\infty} U_n f = f$. Hence from Lemma 4.1

$$U_n(f,x) - f(x) = \sum_{k=n+1}^{\infty} \frac{\varphi(x)U_k''(f,x)}{k(k-1)}. \tag{4.6}$$

Now we shall improve the result of (3.10) for the operator U_n and $f \in W_\infty^2(\varphi)\{0;1\}$. From (4.6),Lemma 4.2 and (2.4) we have

$$\|U_n f - f\|_\infty = \Big\| \sum_{k=n+1}^{\infty} \frac{U_k(\varphi f'', \cdot)}{k(k-1)} \Big\|_\infty \le \|\varphi f''\|_\infty \sum_{k=n+1}^{\infty} \frac{1}{k(k-1)} = \frac{1}{n}\|\varphi f''\|_\infty.$$

For $f_0(x)$ (which is not in $W_\infty^2(\varphi)\{0;1\}$) we have

$$|U_n(f_0,x) - f_0(x)| = \Big| \sum_{k=n+1}^{\infty} \frac{\varphi(x)U_k''(f_0,x)}{k(k-1)} \Big|.$$

Using the idea of the proof of Lemma 4.2 we get $\varphi(x)U_k''(f_0,x) = U_k(\varphi f_0'', x) - x^k - (1-x)^k$ and then

$$\begin{aligned}|U_n(f_0,x) - f_0(x)| &= \Big| \sum_{k=n+1}^{\infty} \frac{U_k(\varphi f_0'', x) - x^k - (1-x)^k}{k(k-1)} \Big| \\ &= \Big| \frac{1}{n} - \sum_{k=n+1}^{\infty} \frac{x^k + (1-x)^k}{k(k-1)} \Big|.\end{aligned}$$

Therefore

$$\|U_n f_0 - f_0\|_\infty = \frac{1}{n} - \sum_{k=n+1}^{\infty} \frac{1}{2^{k-1}k(k-1)}.$$

Thus

$$\frac{1}{n}(1 - \frac{1}{2^n}) \le \|U_n f_0 - f_0\|_\infty \le \frac{1}{n}. \tag{4.7}$$

From Lemma 3.2 and (4.7) we get

$$\|U_n f - f\|_\infty \le \frac{1}{n}\|\varphi f''\|_\infty \tag{4.8}$$

for every $f \in W^2_\infty(\varphi)$. The inequality of Jackson-type (4.8) is better than (3.10).

Theorem 4. *The constant 1 is the minimal in (4.8).*

Proof. If we suppose that, there exists $c \in (0,1)$ such that

$$\|U_n f - f\|_\infty \le \frac{1-c}{n}\|\varphi f''\|_\infty \tag{4.9}$$

for every $f \in W^2_\infty(\varphi)$ and every $n \in N$. In particular for $f = f_0$

$$\|U_n f_0 - f_0\|_\infty \le \frac{1-c}{n}. \tag{4.10}$$

From the first inequality in (4.7) and (4.10) we have $2^{-n} \ge c$ for every n. Therefore (4.9) leads to a contradition. Hence, Theorem 4 is proved.

5. Inverse theorem

In order to show the inverse theorem we shall prove the following two lemmas

Lemma 5.1. *For every $f \in S$ we have*

$$\|U_n f - f - \frac{1}{n}\varphi f''\|_\infty \le \frac{1}{2n^2}\|\varphi(\varphi f'')''\|_\infty,$$

where

$$S = \{f \in W^2_\infty(\varphi)\{0;1\}; \varphi f'' \in W^2_\infty(\varphi)\}.$$

Proof. From (4.6), Lemma 4.2 and (4.8) we have

$$\|U_n f - f - \frac{1}{n}\varphi f''\|_\infty = \Big\| \sum_{k=n+1}^{\infty} \frac{U_k(\varphi f'') - \varphi f''}{k(k-1)} \Big\|_\infty$$

$$\leq \|\varphi(\varphi f'')''\|_\infty \sum_{k=n+1}^{\infty} \frac{1}{k^2(k-1)}.$$

But

$$\sum_{k=n+1}^{\infty} \frac{1}{k^2(k-1)} < \sum_{k=n+1}^{\infty} \frac{(n+2)k}{(n+1)(k+1)} \frac{1}{k^2(k-1)}$$

$$= \frac{(n+2)}{(n+1)} \sum_{k=n+1}^{\infty} \frac{1}{k(k^2-1)}.$$

Hence

$$\sum_{k=n+1}^{\infty} \frac{1}{k^2(k-1)} < \frac{n+2}{n+1} \frac{1}{2n(n+1)} < \frac{1}{2n^2}.$$

Therefore Lemma 5.1 is proved.

Lemma 5.2. *For every $f \in L_\infty[0,1]$ and every $n \in N$ we have*

$$\|\varphi U_n'' f\|_\infty \leq \sqrt{2} n \|f\|_\infty.$$

Proof. Using (4.1) we get

$$\frac{|U_n''(f,x)\varphi(x)|}{n} = \Big| \sum_{k=0}^{n} u_{n,k}(f) P_{n,k}(x) \Big(\frac{n(\frac{k}{n}-x)^2}{\varphi(x)} - 1 - \frac{(1-2x)(\frac{k}{n}-x)}{\varphi(x)} \Big) \Big|$$

$$\leq \|f\|_\infty \sum_{k=0}^{n} P_{n,k}(x) \Big| \frac{n(\frac{k}{n}-x)^2}{\varphi(x)} - 1 - \frac{(1-2x)(\frac{k}{n}-x)}{\varphi(x)} \Big|.$$

The Cauchi-Shwarz inequality gives

(5.1)

$$\frac{|U_n''(f,x)\varphi(x)|}{n} \leq \|f\|_\infty \Big[B_n \Big(\Big(\frac{n(t-x)^2}{\varphi(x)} - 1 - \frac{(1-2x)(t-x)}{\varphi(x)} \Big)^2, x \Big) B_n(1,x) \Big]^{\frac{1}{2}}$$

$$= \|f\|_\infty \Big[\frac{n^2}{\varphi^2(x)} B_n((t-x)^4, x) + 1 + \frac{(1-2x)^2}{\varphi^2(x)} B_n((t-x)^2, x)$$

$$- \frac{2n}{\varphi(x)} B_n((t-x)^2, x) - \frac{2n(1-2x)}{\varphi^2(x)} B_n((t-x)^3, x)$$

$$+ \frac{2(1-2x)}{\varphi(x)} B_n(t-x, x) \Big]^{\frac{1}{2}}.$$

The following properties of Bernstein polinomials are valid [6, p.14]

$$B_n(1,x)=1, \tag{5.2}$$

$$B_n(t-x,x)=x, \tag{5.3}$$

$$B_n((t-x)^2,x)=\frac{\varphi(x)}{n}, \tag{5.4}$$

$$B_n((t-x)^3,x)=-\frac{2x\varphi(x)}{n^2}+\frac{\varphi(x)}{n^2}, \tag{5.5}$$

$$B_n((t-x)^4,x)=3\frac{(n-2)\varphi^2(x)}{n^3}+\frac{\varphi(x)}{n^3}. \tag{5.6}$$

Replacing (5.2), (5.3), (5.4), (5.5) and (5.6) in (5.1) we recieve

$$\begin{aligned}\frac{|U_n''(f,x)\varphi(x)|}{n} &\le \|f\|_\infty\Bigg[\frac{n^2}{\varphi^2(x)}\Big(3\frac{\varphi^2(x)}{n^2}-6\frac{\varphi^2(x)}{n^3}+\frac{\varphi(x)}{n^3}\Big)+1\\ &\quad+\frac{(1-2x)^2}{n\varphi(x)}-2-\frac{2n(1-2x)}{\varphi^2(x)}\Big(-\frac{2x\varphi(x)}{n^2}+\frac{\varphi(x)}{n^2}\Big)\Bigg]^{\frac{1}{2}}\\ &=\|f\|_\infty\sqrt{2-\frac{2}{n}}\end{aligned}$$

and therefore

$$\frac{1}{n}\|\varphi U_n''f\|_\infty\le\sqrt{2}\|f\|_\infty.$$

Lemma 5.2 is proved.

Theorem 5. (Inverse theorem of type A). *For every $f\in L_\infty[0,1]$ we have*

$$K\Big(f,\frac{1}{n}\Big)_\infty\le(6+\sqrt{8})\|U_nf-f\|_\infty.$$

Proof. Using Lemma 5.1 and Lemma 4.2 with U_n^2f instead of f we derive

$$\|U_n^3f-U_n^2f-\frac{1}{n}\varphi(U_n^2f)''\|_\infty\le\frac{1}{2n^2}\|\varphi(\varphi(U_n^2f)'')''\|_\infty=\frac{1}{2n^2}\|\varphi U_n''(\varphi U_n''f)\|_\infty. \tag{5.7}$$

From (5.7) and Lemma 5.2 we have that

$$\begin{aligned}\|U_n^3f-U_n^2f-\frac{1}{n}\varphi(U_n^2f)''\|_\infty &\le\frac{\sqrt{2}}{2n}\|\varphi U_n''f\|_\infty\\ &\le\frac{\sqrt{2}}{2n}\|\varphi(U_n^2f)''\|_\infty+\frac{\sqrt{2}}{2n}\|\varphi U_n''(f-U_nf)\|_\infty\\ &\le\frac{\sqrt{2}}{2n}\|\varphi(U_n^2f)''\|_\infty+\|U_nf-f\|_\infty.\end{aligned}$$

Using this inequality and (2.4) we have

$$\frac{1}{n}\|\varphi(U_n^2 f)''\|_\infty \le \|U_n^3 f - U_n^2 f - \frac{1}{n}\varphi(U_n^2 f)''\|_\infty + \|U_n^3 f - U_n^2 f\|_\infty$$
$$\le 2\|U_n f - f\|_\infty + \frac{\sqrt{2}}{2n}\|\varphi(U_n^2 f)''\|_\infty.$$

Hence

$$\frac{(2-\sqrt{2})}{2n}\|\varphi(U_n^2 f)''\|_\infty \le 2\|U_n f - f\|_\infty,$$
$$\frac{1}{n}\|\varphi(U_n^2 f)''\|_\infty \le (4+2\sqrt{2})\|U_n f - f\|_\infty.$$

Therefore

$$K\left(f,\frac{1}{n}\right)_\infty \le \|U_n^2 f - f\|_\infty + \frac{1}{n}\|\varphi(U_n^2 f)''\|_\infty$$
$$\le \|U_n f - f\|_\infty + \|U_n^2 f - U_n f\|_\infty + (4+2\sqrt{2})\|U_n f - f\|_\infty,$$
$$\le (6+\sqrt{8})\|f - U_n f\|_\infty.$$

Theorem 5 is proved.

Theorem 3.2 and Theorem 5 yield (1.1).

Remark. If we look trough the relation between $\|f - U_n f\|_\infty$ and $K\left(f,\frac{1}{2n}\right)_\infty$ we can prove a better estimate

$$\frac{1}{2}\|U_n f - f\|_\infty \le K\left(f,\frac{1}{2n}\right)_\infty \le (4+\sqrt{2})\|U_n f - f\|_\infty$$

in a sence that it has the smaller quotient c_2/c_1 for the constants in (1.2).

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