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Locally k–Nearly Uniformly Convex Banach Spaces

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Presented by Bl. Sendov

The class of locally k-nearly uniform by convex Banach spaces is introduced. It is proved that the class lies strictly between the class of locally k-uniformly rotund Banach spaces and the class of locally fully k-convex Banach spaces.

Different uniform geometrical properties have been defined between uniform convexity and reflexivity of Banach spaces. Sullivan [17] defined k-uniformly rotund $(k-UR)$ Banach spaces. Fan and Glicksberg [1] introduced the fully k-convex (kR) Banach spaces. In [11], it is proved that every strictly convex $k-UR$ spaces is $(k+1)R$. Another uniform property is the nearly uniform convexity (NUC), introduced by Huff [3].

Recall that the Kuratowski measure of non-compactness $\alpha(A)$ of a set A in a Banach space X is the infimum of those $\varepsilon > 0$ for which there is a covering of A by a finite number of sets each has diameter less than ε.

Let X be a Banach space with closed unit ball B. By the drop $D(x, B)$ defined by an element $x \in X \setminus B$, we mean the convex hull, $co(\{x\} \cup B)$, of x and B. Let $R(x,B) = D(x,B) \setminus B$. X is said to have the property (β) [16] if for each $\varepsilon > 0$, there is $\delta > 0$, such that $1 < \|x\| < 1+\delta$ implies $\alpha(R(x,B)) < \varepsilon$. It follows from [15] and [16] that property (β) is isomorphically between uniform convexity and nearly uniform convexity.

In [7], new uniform convexity $k\beta$ and $k-NUC$, $k \geq 2$, have been introduced, where $1-\beta$ coincides with (β). It is proved that for all $k \geq 1$, $k-\beta$ implies $(k+1)-NUC$ and that for $k \geq 2$, $k-NUC$ implies $k-\beta$. Moreover, an example of an $8-NUC$ space is given which fails to have an equivalent $1-\beta$ norm. From the definitions, it follows that NUC is the asymptotic property of both $k-NUC$ and $k-\beta$. Furthermore, it is known that $k-UR$ implies $k-\beta$ and if X is strictly convex and $k-NUC$, then X is kR, which improves isomorphically, the result in [11].

Nan and Wang [12] defined locally k-uniformly rotund ($Lk-UR$) and locally fully k-convex (LkR) Banach spaces. They prove every strictly convex $Lk-UR$ space is LkR. They have also pointed out that $L1R = L1-UR = LUR$, locally uniform convexity, but for every $k \geq 2$, LkR and $Lk-UR$ are different properties.

In the paper, we study the local versions, $Lk-\beta$, $Lk-NUC$ and $LNUC$ of $k-\beta$, $k-NUC$ and NUC. Although $k-\beta$ and $k-NUC$ are distinct [7], we show that $Lk-\beta$ and $Lk-NUC$ coincide and we shall use the notion $Lk-NUC$. $LNUC$ is the asymptotic property of $Lk-NUC$ and several characterizations of $LNUC$ are given. We also improve the results of [12]. Finally, examples are given to distinguish $Lk-NUC$ from $Lk-UR$ and LkR.

Throughout this paper, let B denote the closed unit ball of a Banach space X and S^* denotes the unit sphere in the dual space X^*. For a sequence $\{x_n\}$ in X, the separation constant of $\{x_n\}$ is defined by, $sep(x_n) = \inf\{\|x_n - x_m\| : n \neq m\}$.

Definition. A Banach space X is called *locally nearly uniformly convex* ($LNUC$) if for every x in X, $\|x\| = 1$ and for every $\varepsilon > 0$, there is a $\delta = \delta(x,\varepsilon) > 0$ such that for every sequence $\{x_n\}$ in B, $sep(x_n) > \varepsilon$, then $co(\{x\} \cup \{x_n\}) \cap (1-\delta)B \neq \emptyset$.

Using the argument as in [16] and [7], the following can be proved.

Theorem 1. *For any Banach space X, the following are equivalent,*

(i) X is $LNUC$;

(ii) For every x in X, $\|x\| = 1$ and for each $\varepsilon > 0$, there is a $\delta > 0$ such that for all f in S^*, if $x \in S(f,\delta)$ then $\alpha(S(f,\delta)) < \varepsilon$;

(iii) For every x in X, $\|x\| = 1$,

$$\lim_{t\to 1+} \sup\{\alpha(C) : C \subset R(tx, B), C convex\} = 0$$

Definition. Let $k \geq 1$ be an integer. A Banach space X is said to be *locally k-nearly uniformly convex* ($Lk-NUC$) if for each x, $\|x\| = 1$ and each $\varepsilon > 0$, there is a $\delta = \delta(x,\varepsilon) > 0$ such that for all sequence $\{x_n\}$ in B, $sep(x_n) > \varepsilon$, then there exist $(n_1, \ldots, n_k)$ such that $\frac{1}{k+1}\|x + \sum_{i=1}^k x_{n_i}\| \leq 1-\delta$.

Recall that X is k-nearly uniformly convex [7] if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all sequence $\{x_n\}$ in B, $sep(x_n) > \varepsilon$, there exists $(n_1, \ldots, n_k)$ such that $\frac{1}{k}\|\sum_{i=1}^{k} x_{n_i}\| \leq \delta$. It is clear that for all k, every $Lk - NUC$ spaces is $LNUC$.

A Banach space X is $k-\beta$ [7] if for each $\varepsilon > 0$, there is a δ, $0 < \delta < 1$, such that for all x in B and for any sequence $\{x_n\}$ in B, $sep(x_n) > \varepsilon$, then there exist $(n_1, \ldots, n_k)$ such that $co(x, x_{n_1}, \ldots, x_{n_k}) \cap (1-\delta)B \neq \emptyset$. It follows from Theorem 2 below that the class of locally $k-\beta$ spaces, as defined by condition (iii) of Theorem 2, coincides with the class of locally $k - NUC$ spaces. As in [7], the following can be proved.

Theorem 2. *Let $k \geq 1$ and X be a Banach space. Then the following are equivalent.*

(i) X is $Lk - NUC$

(ii) For every x, $\|x\| = 1$ and every $\varepsilon > 0$, there is a $\delta > 0$, such that for any sequence $\{x_n\}$ in B, $sep(x_n) > \varepsilon$, there are $(n_1, \ldots, n_k)$, such that $co(x, x_{n_1}, \ldots, x_{n_k}) \cap (1-\delta)B \neq \emptyset$,

(iii) For every x, $\|x\| = 1$,

$$\lim_{t\to 1+} \sup\{\alpha(C) \colon C \subset R(tx, B), co(x_1, \ldots, x_k) \subset R(tx, B)$$

$$\textit{for all } (x_1, \ldots, x_k) \textit{ in } C\} = 0.$$

A Banach space X is said to be an $Lk-UR$ space [12] if for every x, $\|x\| = 1$ and every $\varepsilon > 0$, there is a $\delta = \delta(x, \varepsilon) > 0$ such that for any x_i, $\|x_i\| = 1$, $i = 1, \ldots, k$, $\|x + x_1 + \ldots + x_k\| > k + 1 - \delta(x, \varepsilon)$ implies $V(x, x_1, \ldots, x_k) < \varepsilon$ where

$$V(u_1, \ldots, u_{k+1}) = \sup\left\{\begin{vmatrix} 1 & \ldots & 1 \\ f_1(u_1) & & f_1(u_{k+1}) \\ \ldots & \ldots & \ldots \\ f_k(u_1) & \ldots & f_k(u_{k+1}) \end{vmatrix} : \begin{array}{c} f_i \in S^* \\ i = 1, \ldots, k \end{array}\right\}.$$

X is said to be an LkR space if for any x, $\|x\| = 1$ and any sequence $\{x_n\}$ such that $\lim_{n_1, \ldots, n_k \to \infty} \|x + x_{n_1} + \ldots + x_{n_k}\| = k + 1$ and $\lim_n \|x_n\| = 1$ then $\lim_n \|x_n - x\| = 0$. It was proved in [12] that strictly convex $Lk - UR$ spaces are LkR. Indeed, the following is true.

Theorem 3. *Let $k \geq 1$ be an integer and let X be a Banach space. Then (i) If X is an $Lk - UR$ space then X is $Lk - NUC$;*

(ii) If X is strictly convex and $Lk-NUC$ then X is LkR.

Proof. (i) Suppose that there exists x, $\|x\|=1$ and $\varepsilon>0$ such that for every integer m, there is a sequence $\{x_n^{(m)}\}$ in B, $sep(x_n^{(m)})>\varepsilon$ and for any $(n_1,\ldots,n_k)$, $\frac{1}{k+1}\|x+\sum_{i=1}^k x_{n_i}^{(m)}\|>1-\frac{1}{m}$. Since X is $Lk-UR$, it follows that $\lim_m V\left(x,x_{n_1}^{(m)},\ldots,x_{n_k}^{(m)}\right)=0$.

On the other hand, without loss of generality, we may assume that $\|x_n^{(m)}\|=1$ for all n,m. Since $sep\left(x_n^{(m)}\right)>\varepsilon$, we may find inductively $\{x_{n_i}^{(m)}\}_{i=1,\ldots,k}$ such that for all $i=1,\ldots,k$, $dist\left(x_{n_i}^{(m)},aff(x,x_{n_1}^{(m)},\ldots,x_{n_{i-1}}^{(m)})\right)>\frac{\varepsilon}{2}$ where $aff\left(x,x_{n_1}^{(m)},\ldots,x_{n_{i-1}}^{(m)}\right)$ is the affine hull of $\{x,x_{n_1}^{(m)},\ldots,x_{n_{i-1}}^{(m)}\}$. Then it follows from [2] that for all $m=1,2,\ldots$, $V\left(x,x_{n_1}^{(m)},\ldots,x_{n_k}^{(m)}\right)>(\frac{\varepsilon}{2})^k$ which is a contradiction.

(ii) Let $\|x\|=1$ and $\{x_n\}$ be a sequence such that

$$(*)\qquad \lim_{n,\ldots,n_k\to\infty}\frac{1}{k+1}\|x+\sum_{i=1}^k x_{n_i}\|=1$$

We may assume that $\|x_n\|=1$, $n\in\mathbb{N}$. Suppose that there is a subsequence $\{x_j\}$ of $\{x_n\}$ without any Cauchy subsequence. Then there is an $\varepsilon>0$ and a subsequence $\{x_m\}$ of $\{x_j\}$ with $sep(x_m)>\varepsilon$. Since X is $Lk-NUC$, there is a $\delta>0$ such that $\frac{1}{k+1}\|x+\sum_{i=1}^k x_{n_i}\|\le 1-\delta$ for arbitrary large $(n_1,\ldots,n_k)$ which is impossible by $(*)$. Thus every subsequence of $\{x_n\}$ has a Cauchy subsequence. Let y be any cluster point of $\{x_n\}$. By $(*)$, we have $\|x+ky\|=k+1$. Since $\|x\|=\|y\|=1$ and X is strictly convex, it follows that $x=y$ and $\{x_n\}$ converges to x.

Remark. It was proved in [9] that a Banach space X is LUR if and only if X is strictly convex and $L-\beta$. This is a consequence of Theorem 3, because $LUR=L1R=L1-UR$ and $L-\beta=L1-\beta=L1-NUC$.

We now give a list of examples to distinguish $Lk-NUC$, $Lk-UR$, LkR and $LNUC$.

Example 1. For each $k\ge 2$, there is a $Lk-NUC$ space which is not $L(k-1)-NUC$.

In [12], for each $k\ge 2$, an example is given of a strictly convex $k-UR$ space X which is not $L(k-1)R$. Hence, by Theorem 3, X is $Lk-NUC$ but is not $L(k-1)-NUC$.

Example 2. There exists a Banach space X with property (β), hence X is $2-NUC$ but X is not $Lk-UR$ for all $k \geq 1$.

Let $X = [\sum_{n=1}^{\infty} \oplus l_1^n]_{l_2}$. By [4], X has property (β). To see that X is not $Lk-UR$ for all $k \geq 1$, fix k and consider

$$x_i = (\underbrace{0, ..., 0}_{k+1}, e_i, 0, ...), \quad i = 1, 2, ..., k+1$$

where $\{e_1, \ldots, e_n\}$ is the usual basis of l_1^n. Obviously, $\| \sum_{i=1}^{k+1} x_i \| = k+1$, but $V(x_1, \ldots, x_{k+1}) > 1$. Thus X is not $Lk-UR$.

Example 3. For each $k \geq 2$, there is a strictly convex $2-NUC$ space which is not $Lk-UR$.

Let $E = (l_2, ||| \cdot |||)$ where for $x = (a_1, a_2, ...,) \in E$,

$$|||x|||^2 = \{|a_1| + (a_2^2 + a_3^3 + ...)^{1/2}\}^2 + \{\left(\frac{a_2}{2}\right)^2 + ... + \left(\frac{a_n}{n}\right)^2 + ...\}$$

The space E was studied in [12], [11], [14] and [17].

Let X_k be the l_2^{k+1}-sum of E. It is clear that X_k is strictly convex and it follows from [12] that X_k is not $Lk-UR$. Since $2-NUC$ is preserved by finite l_2-sums [8], it remains to show that E is $2-NUC$.

Let $x = (a_1, a_2, ...,) \in E$. For convenience, let us denote $qx = a_1$, $px = (0, a_2, a_3, ...,)$ and $Tx = \left(0, \frac{a_2}{2}, ..., \frac{a_n}{n}, ...\right)$. Let $\| \cdot \|$ be the usual norm in l_2. Then $|||x|||^2 = \{|qx| + \|px\|\}^2 + \|Tx\|^2$ for all x in E.

Given $\varepsilon > 0$, by uniform convexity of l_2, there is δ_1, $0 < \delta < \frac{1}{2}$, such that for y_1, y_2 in l_2, $\|y_1\| = \|y_2\| = 1$, if $\|y_1 - y_2\| > \frac{\varepsilon}{3}$, then $\frac{1}{2}\|y_1 + y_2\| < 1 - 2\delta_1$. Put $\delta = \frac{\varepsilon^2 \delta_1}{16}$. Let $\{x_n\}$ be any sequence with $|||x_n||| \leq 1$ and $sep(x_n) > \varepsilon$ in E. Passing to a sequence, we may assume that $qx_n \to b_1$, $\|px_n\| \to b_2$ and $\|Tx_n\| \to b_3$. Clearly $(|b_1| + b_2)^2 + b_3^2 \leq 1$. Since $\|Tx\| \leq \|px\|$ and $\{qx_n\}$ is convergent, it follows from $sep(x_n) > \varepsilon$ that if we consider n and m greater than some fixed number, we shall have $\|px_n - px_m\| > \frac{\varepsilon}{2}$ for $n \neq m$. Therefore $b_2 \geq \frac{\varepsilon}{4}$ and moreover, for n and m sufficiently large, we have $\frac{1}{2}\|px_n + px_m\| < (1-\delta_1)b_2$. Let $\eta < \frac{1}{64}\delta_1^2\varepsilon^2$. Then for sufficiently large n, m, $n \neq m$,

$$|||\frac{1}{2}(x_n + x_m)|||^2 \leq \{|b_1| + \eta + (1 - \delta_1)b_2\}^2 + (b_3 + \eta)^2$$

$$\leq 1 - 2b_2^2(\delta_1 - \delta_1^2) \leq 1 - \frac{\delta_1 \varepsilon^2}{16} = 1 - \delta.$$

This completes the proof that E is $2-NUC$.

For the remaining examples, we need the following.

Theorem 4. *Let $(X, \|\cdot\|)$ be a $2R$ space with normalized basis $\{e_n\}$. Define for all $x = \sum_{n=1}^{\infty} a_n e_n$ in X,*

$$|||x||| = \left\{ \left(|a_1| + \|\sum_{n=2}^{\infty} a_n e_n\| \right)^2 + \sum_{n=2}^{\infty} (\frac{a_n}{n})^2 \right\}^{1/2}.$$

Then $(X, |||\cdot|||)$ is a $2R$ Banach space.

Proof. It is easy to see that $(X, |||\cdot|||)$ is a Banach space.

As in Example 3, for $x = \sum_{n=1}^{\infty} a_n x_n$ in X, let $qx = a_1$, $px = \sum_{n=2}^{\infty} a_n e_n$ and $T\colon X \to l_2$ defined by $Tx = (\frac{a_2}{2}, ..., \frac{a_n}{n}, ...)$. Let also $\tau(x) = |qx| + \|px\|$ and $s(x) = \|Tx\|_2$ where $\|\cdot\|_2$ is the usual norm in l_2. We shall follow the proof in [14].

Let $\{x_n\}$ be a sequence in X such that $\lim_{n,m\to\infty} |||x_n^+ + x_m||| = 2$. Therefore,

$$\begin{aligned} |||x_n + x_m||| &= \|(\tau(x_n + x_m), s(x_n + x_m)\|_2 \\ &\le \|(\tau x_n, s x_n) + (\tau x_m, s x_m)\|_2 \\ &\le |||x_n||| + |||x_m||| \to 2. \end{aligned}$$

Since l_2^2 is $2R$, $\tau x_n \to r_0$ and $s x_n \to s_0$ for some r_0 and s_0. It follows from the above inequalities that $\tau(x_n + x_m) \to 2r_0$ and $s(x_n + x_m) \to 2s_0$.

Clearly $|||\cdot|||$ is strictly convex, hence $\{x_n\}$ has unique cluster point. To show that $\{x_n\}$ is convergent, it remains to show that every subsequence of $\{x_n\}$ has a convergent subsequence. Let $\{x_j\}$ be any subsequence of $\{x_n\}$. Passing to a subsequence, we may assume that $qx_j \to q_0$. Thus $\|px_j\| \to r_0 - |q_0|$ and $\|p(x_i + x_j)\| \to 2(r_0 - |q_0|)$ as $i, j \to \infty$. Obviously, this also implies $s(x_i - x_j) \to 0$. Thus $\{x_j\}$ is convergent, which means that $|||\cdot|||$ is $2R$.

Example 4. There exists a $2R$ space which is not $LNUC$.

Let $(X, \|\cdot\|)$ be the l_2-sum of $\{l_n, n \ge 2\}$. Then $(X, \|\cdot\|)$ is $2R$. Consider $(X, |||\cdot|||)$ as in Theorem 4. It follows that $(X, |||\cdot|||)$ is $2R$. We claim that $(X, |||\cdot|||)$ is not $LNUC$.

According to [3], X does not have any equivalent NUC norm, in particular, $\|\cdot\|$ is not NUC. Thus, there is an $\varepsilon > 0$ and sequence $\{x_n^{(m)}\}_n$ with $\|x_n^{(m)}\| \le 1$, $sep(x_n^{(m)}) > \varepsilon$ in $(X, \|\cdot\|)$ but $\|y\| > 1 - \frac{1}{m}$ for every y in $co(\{x_n^{(m)}\}_n)$. Without loss of generality, we may assume that $\sup_n |||x_n^{(m)}||| \to 1$ as $m \to \infty$ and the separation constants of $\{x_n^{(m)}\}$ are also greater than ε. We now show that $(X, |||\cdot|||)$ fails to be $LNUC$ at $x = e_1$.

For any $\lambda_i \geq 0$, $\sum_{i=0}^k \lambda_i = 1$ and any $(n_1, \ldots, n_k)$, we have

$$|||\lambda_0 x + \sum_{i=1}^k \lambda_i x_{n_i}^{(m)}|||^2 = (\lambda_0 + \|\sum_{i=1}^k \lambda_i x_{n_i}^{(m)}\|)^2 + \left(s(\sum_{i=1}^k \lambda_i x_{n_i}^{(m)})\right)^2$$
$$\geq \left(\lambda_0 + (1-\lambda_0)(1-\frac{1}{m})\right)^2 \geq (1-\frac{1}{m})^2 \to 1.$$

This completes the proof that $(X, |||\cdot|||)$ is not $LNUC$.

Example 5. There exists a $LNUC$ space which is not $Lk-NUC$ for all k.

Let $(X, \|\cdot\|)$ be the Baernstain's space with the equivalent $2R$ norm $\|\cdot\|$ defined in [10]. Consider $(X, |||\cdot|||)$ as in Theorem 4. We first observe that $(X, |||\cdot|||)$ is, in fact, NUC.

For any x and y with $\max\{i \in \operatorname{supp} x\} < \min\{i \in \operatorname{supp} y\}$, we obviously have $|||x+y|||^2 \geq |||x|||^2 + |||y|||^2$, which implies that $(X, |||\cdot|||)$ is NUC (see the proof of Theorem 3 in [13]).

Now, fix $k \geq 1$. For each $m \in \mathbb{N}$, let $X_m = \{\sum_{i=m}^\infty a_i e_i \colon \sum_{i=1}^\infty a_i e_i \in X\}$. Then $(X_m, \|\cdot\|)$ has a spreading model equivalent to l_1 [10]. Hence there is a bounded sequence $\{x_n^{(m)}\}_n$ in X_m so that for all j, if $j \leq n_1 < \ldots < n_{2^j}$, then for all $c_1, \ldots, c_{2^j}$, we have

$$(1-\frac{1}{m})\sum_{i=1}^{2^j} |c_i| \leq \|\sum_{i=1}^{2^j} c_i x_{n_i}^{(m)}\| \leq (1+\frac{1}{m})\sum_{i=1}^{2^j} |c_i|.$$

Take l so that $2^l \geq k$. Then for every $n \geq l$, $1 - \frac{1}{m} \leq \|x_n^{(m)}\| \leq 1 + \frac{1}{m}$. Moreover, if $i \neq j$, $i, j \geq l$, $\|x_i^{(m)} - x_j^{(m)}\| \geq 2(1-\frac{1}{m})$. Thus the separation constant of $\{x_n^{(m)}\}_{n \geq l}$ with respect to $|||\cdot|||$ is also greater than 1. Furthermore, if $l \leq n_1 < \ldots < n_k$, then $\|\sum_{i=1}^k x_{n_i}^{(m)}\| \geq k(1-\frac{1}{m})$. Denote $x = e_1$. Then for any $\{n_i\}_{i=1}^k$ with $l \leq n_1 < \ldots < n_k$,

$$|||x + \sum_{i=1}^k x_{n_i}^{(m)}|||^2 \geq \left(1 + \|\sum_{i=1}^k x_{n_i}^{(m)}\|\right)^2 \geq \left(1 + k(1-\frac{1}{m})\right)^2 \to (k+1)^2$$

as $m \to \infty$. Also, by the choice of $\{x_n^{(m)}\}_n$, $\sup_n s(x_n^{(m)}) \to 0$ as $m \to \infty$, whence $\sup_n |||x_n^{(m)}||| \to 1$. This completes the proof that $(X, |||\cdot|||)$ is not $Lk-NUC$.

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