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A Unified Approach to Certain Fractional Integration Operators

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In order to unify and extend the results on various fractional integration operators defined and studied during the last three decades, the authors investigate in this paper a pair of new fractional integral operators associated with multivariable H-function due to H. M. Srivastava and R. Panda [23] and a general class of polynomials introduced by Srivastava [22]. Some properties of these operators related to their Mellin transforms are investigated, generalizing the scores of results on various fractional integration operators scattered in literature.

1. Introduction

The object of this paper is to introduce a pair of new fractional integration operators and to establish some theorems for these operators involving their Mellin transforms.

These operators are extensions of the operators of fractional integration defined and studied earlier by many authors, e.g. by A. Erdélyi [2], H. Kober [10], R.K. Saxena [15], R.K. Saxena and R.K. Kumbhat [16,17,18], S.L. Kalla [4,5], S.L. Kalla and R.K. Saxena [6], S.L. Kalla and V. Kiryakova [7,8], V. Kiryakova [9], M. Saigo [12], M. Saigo et al. [13,14], E.R. Love [11], R.K. Saxena and G.C. Modi [19,20], S.P. Goyal et al. [3], R.K. Saxena and V. Kiryakova [21], etc.

The multivariable H-function introduced and studied by H.M. Srivastava and R. Panda [23]; K.C. Gupta and S.P. Goyal [24, p.251, Eq. (c.1) to (c.3)] will be defined and represented in the following manner:

$$H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = H^{O,N;M_1,N_1;\ldots;M_r,N_r}_{P,Q;P_1,Q_1;\ldots;P_r,Q_r}$$

$$\tag{1.1} \times \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)})_{1,P} : (c_j,\gamma_j^{(1)})_{1,P_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,P_r} \\ (b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)})_{1,Q} : (d_j,\delta_j^{(1)})_{1,Q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,Q_r} \end{matrix}\right]$$

$$= \frac{1}{(2\pi i)^r}\underbrace{\int_{L_1}\cdots\int_{L_r}}_{(r)} \Phi_1(s_1)\ldots\Phi_r(s_r)\Psi(s_1,\ldots,s_r)\cdot z_1^{s_1}\ldots z_r^{s_r}ds_1\ldots ds_r,$$

where

$$\Phi_i(s_i) = \frac{\prod_{j=1}^{M_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{N_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod_{j=M_i+1}^{Q_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=N_i+1}^{P_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}s_i)}, \quad (i=1,\ldots,r)$$

and

$$\Psi(s_1,\ldots,s_r) = \frac{\prod_{j=1}^{N}\Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i)}{\prod_{j=N+1}^{P_i}\Gamma(a_j-\sum_{j=1}^{r}\alpha_j^{(i)}s_i)\prod_{j=1}^{Q_i}\Gamma(1-b_j+\sum_{j=1}^{r}\beta_j^{(i)}s_i)}.$$

For the conditions of existence on the various parameters of the multivariable H-function, we refer the reader to H. M. Srivastava et al. [24, pp. 251-253, Eqns (c.2) to (c.8)].

The general class of polynomials denoted by $S_{n'}^{m'}[x]$, due to Srivastava [22, p.1, Eqn (1)], is defined in the following form:

$$\tag{1.2} S_{n'}^{m'}[x] = \sum_{k'=0}^{[n'/m']}\frac{(-n')_{m'k'}}{k'!}A_{n',k'}x^{k'},$$

where m', n' are arbitrary positive integer and the coefficients $A_{n',k'}$ $(n',k'\geq 0)$ are arbitrary real or complex constants.

2. Definitions

A pair of new generalized fractional integration operators are defined by means of the following equalities:

$$\tag{2.1} I^{\delta,\beta}_{\lambda_n}[f(x)] = \xi x^{-\delta-\xi\beta-1}\int_o^x t^\delta(x^\xi-t^\xi)^\beta H\begin{bmatrix}\lambda_1 U \\ \vdots \\ \lambda_n U\end{bmatrix} \times\prod_{i=1}^{r} S^{m_i'}_{n_i'}\left[z_i(\frac{t^\xi}{x^\xi})^{e_i}(1-\frac{t^\xi}{x^\xi})^{f^i}\right]\Phi\left[\frac{t^\xi}{x^\xi}\right]f(t)dt,$$

$$R_{\lambda_n}^{\alpha,\beta}[f(x)] = \xi x^{\alpha} \int_o^{\infty} t^{-\alpha-\xi\beta-1}(t^{\xi}-x^{\xi})H\begin{bmatrix}\lambda_1 V\\ \vdots \\ \lambda_n V\end{bmatrix} \times \prod_{i=1}^{r} S_{n_i'}^{m_i'}\left[z_i(\frac{t^{\xi}}{x^{\xi}})^{e_i}(1-\frac{x^{\xi}}{t^{\xi}})^{f^i}\right]\Phi\left[\frac{x^{\xi}}{t^{\xi}}\right]f(t)dt, \tag{2.2}$$

where U and V respectively stand for the expressions:

$$U = (\frac{t^{\xi}}{x^{\xi}})^{M_i}(1-\frac{t^{\xi}}{x^{\xi}})^{N_i},$$

$$V = (\frac{x^{\xi}}{t^{\xi}})^{M_i}(1-\frac{x^{\xi}}{t^{\xi}})^{N_i},$$

and ξ, M_i, N_i are positive numbers. The kernel $\Phi(\frac{t^{\xi}}{x^{\xi}})$ occurring in (2.1) and (2.2) is supposed to be a continuous function such that the integrals make sense for wide classes of functions $f(x)$.

These operators exist under the following set of conditions:

(i) $$1 \le p,q < \infty, p^{-1}+q^{-1} = 1,$$

(ii) $$\Re(\delta + \xi\sum_1^r M_i \frac{b_j^{(i)}}{\beta_j^{(i)}}) > -\frac{1}{q},$$

(iii) $$\Re(\beta + \xi\sum_1^r N_i \frac{b_j^{(i)}}{\beta_j^{(i)}}) > -\frac{1}{q},$$

(iv) $$\Re(\alpha + \xi\sum_1^r M_i \frac{b_j^{(i)}}{\beta_j^{(i)}}) > -\frac{1}{p},$$

(v) $$f(x) \in L_p(o,\infty),$$

where $j = 1,2,...,M_r$; $i = 1,2,...,r$.

The last condition ensures that $I_{\lambda_n}^{\delta,\beta}[f(x)]$ and $R_{\lambda_n}^{\alpha,\beta}[f(x)]$ both exist and belong to $L_p(0,\infty)$.

3. The Mellin transform

The Mellin transform of $f(x)$ will be denoted by $M[f(x)]$, or by $F(s)$. We write $s = p^{-1} + it$, where p and t are real. If $p \geq 1$, $f(x) \in L_p(0, \infty)$, then for

$$(3.1) \qquad p = 1, M[f(x)] = F(s) = \int_o^\infty x^{s-1} f(x) dx$$

and

$$(3.2) \qquad f(t) = \frac{1}{2\pi i} \int_{c-i_\infty}^{c+i_\infty} F(s) x^{-s} ds,$$

under suitable conditions on the parameters and the variables.

For $p > 1$,

$$(3.3) \qquad M[f(x)] = F(s) = \text{l.i.m.} \int_{1/x}^{x} f(x) x^{s-1} dx,$$

where l.i.m. denotes the usual limit in the mean for L_p-spaces.

4. A special case

On account of the most general character of the multivariable H-function, numerous special cases of operators (2.1) and (2.2) can be derived by giving suitable values to the parameters. For the sake of brevity only one interesting special case is given below:

For $\Phi(x) = 1$ and $n_1' = n_2' = ... = n_r' = 0$,
we obtain the operators defined earlier by Banerji and Sethi [1] in the following form:

$$(4.1) \qquad I_{\lambda_n}^{\delta, \beta} [f(x)] = \xi x^{-\delta-\xi\beta-1} \int_o^x t^\delta (x^\xi - t^\xi)^\beta H \begin{bmatrix} \lambda_1 U \\ \vdots \\ \lambda_n U \end{bmatrix} f(t) dt,$$

$$(4.2) \qquad R_{\lambda_n}^{\alpha, \beta} [f(x)] = \xi x^\alpha \int_o^\infty t^{-\alpha-\xi\beta-1} (t^\xi - x^\xi)^\beta H \begin{bmatrix} \lambda_1 V \\ \vdots \\ \lambda_n V \end{bmatrix} f(t) dt,$$

where the conditions of validity follow from those given along with the operators (2.1) and (2.2).

5. Theorems Associated with Mellin Transforms

Theorem 1. *If $f(x) \in L_p(0,\infty)$, $1 \le p \le 2$ [or $f(x) \in M_p(0,\infty)$, $p > 2$], $p^{-1} + q^{-1} = 1$, $\Re\left[\delta + \xi \sum_{i=1}^{r} M_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > -q^{-1}$, $\Re\left[\beta + \xi \sum_{i=1}^{r} N_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > -q^{-1}$ for $j = 1,2,\dots,M_r$, and the integrals involved are absolutely convergent, then the following result holds:*

$$M\left\{I_{\lambda_n}^{\delta,\beta}[f(x)]\right\} = M\{f(x)\}\, R_{\lambda_n}^{\delta - s + 1,\beta}[1], \tag{5.1}$$

where $M_p(0,\infty)$ denotes the class of all functions $f(x)$ of $L_p(0,\infty)$ with $p > 2$, which are inverse Mellin transforms of the functions of $L_q(-\infty,\infty)$.

Proof. From (2.1), it follows that

$$M\left\{I_{\lambda_n}^{\delta,\beta}[f(x)]\right\} = \int_0^\infty x^{s-1}\left\{\xi x^{-\delta-\xi\beta-1}\int_0^\infty t^{\delta}(x^{\xi} - t^{\xi})^{\beta}\right.$$

$$\left.\times H\begin{bmatrix}\lambda_1 U\\ \vdots\\ \lambda_n U\end{bmatrix} \prod_{i=1}^{r} S_{n_i'}^{m_i'}\left[z_i\left(\frac{t^{\xi}}{x^{\xi}}\right)^{e_i}\left(1-\frac{t^{\xi}}{x^{\xi}}\right)^{f^i}\right] \Phi\left(\frac{t^{\xi}}{x^{\xi}}\right) f(t)dt\right\} dx.$$

Interchanging the order of integration, which is permissible under the conditions stated with the theorem, it is seen that

$$M\left\{I_{\lambda_n}^{\delta,\beta}[f(x)]\right\} = \int_0^\infty t^{s-1} f(t)dt \left\{\xi t^{\delta-s+1}\int_t^\infty x^{s-\delta-\xi\beta-2}\right.$$

$$\left.\times (x^{\xi} - t^{\xi})^{\beta} H\begin{bmatrix}\lambda_1 U\\ \vdots\\ \lambda_n U\end{bmatrix} \prod_{i=1}^{r} S_{n_i'}^{m_i'}\left[z_i\left(\frac{t^{\xi}}{x^{\xi}}\right)^{e_i}\left(1-\frac{t^{\xi}}{x^{\xi}}\right)^{f^i}\right] \Phi\left(\frac{t^{\xi}}{x^{\xi}}\right) dx\right\}.$$

Now, (5.1) follows easily from (2.2).

In a similar manner, the following theorem can be established.

Theorem 2. *If $f(x) \in L_p(0,\infty)$, $1 \le p \le 2$ [or $f(x) \in M_p(0,\infty)$, $p > 2$], $p^{-1} + q^{-1} = 1$, $\Re\left[\beta + \xi \sum_1^r N_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > -q^{-1}$; $\Re\left[\alpha + \xi \sum_1^r M_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > -p^{-1}$ for $j = 1,2,\dots,M_r$; and the integrals involved are absolutely convergent, then the following result holds:*

$$M\left\{R_{\lambda_n}^{\alpha,\beta}[f(x)]\right\} = M\{f(x)\}\, I_{\lambda_n}^{\alpha + s - 1,\beta}[1]. \tag{5.2}$$

Theorem 3. *If* $f(x) \in L_p(0,\infty)$; $p^{-1}+q^{-1}=1$, $g(x) \in L_p(0,\infty)$; $\Re\left[\delta+\xi\sum_1^r M_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > \max(p^{-1},q^{-1})$; $\Re\left[\beta+\xi\sum_1^r N_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > 0$; *for* $j=1,2,\ldots,M_r$; *and the integrals involved are absolutely convergent, then the following result holds:*

$$\int_0^\infty g(x) I_{\lambda_n}^{\delta,\beta}[f(x)]\,dx = \int_0^\infty f(x) R_{\lambda_n}^{\delta,\beta}[g(x)]\,dx. \tag{5.3}$$

Proof. Relation (5.3) follows on interpreting it with the help of (2.1) and (2.2).

Theorem 4. *(Inversion formula): If* $f(x) \in L_p(0,\infty)$, $1 \le p \le 2$ [*or* $f(x) \in M_p(0,\infty)$, $p>2$], $p^{-1}+q^{-1}=1$, $\Re\left[\delta+\xi\sum_1^r M_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > -q^{-1}$; $\Re\left[\beta+\xi\sum_1^r N_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > -q^{-1}$ *for* $j=1,2,\ldots,M_r$; *the integrals involved are absolutely convergent and*

$$I_{\lambda_n}^{\delta,\beta}[f(x)] = [g(x)], \tag{5.4,}$$

then the following result holds:

$$f(x) = \int_0^\infty t^{-1}[g(t)][h(x/t)]\,dt. \tag{5.5}$$

Here

$$[h(x)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{-s}}{R[(s)]}ds \tag{5.6}$$

and

$$R[(s)] = R_{\lambda_n}^{\delta-s+1,\beta}[1]. \tag{5.7}$$

Proof. Multiplying both sides of equation (5.4) by x^{s-1}, integrating from 0 to ∞ with respect to x and applying Theorem 1, we find that

$$M[f(x)] = \frac{M[g(x)]}{R[(s)]}.$$

Now the inverse Mellin transform gives

$$\begin{aligned} f(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\frac{M[g(x)]}{R[(s)]}ds \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\frac{1}{R[(s)]}\int_0^\infty t^{s-1}g(t)dtds \\ &= \int_0^\infty t^{-1}[g(t)][h(x/t)]dt. \end{aligned}$$

This completes the proof of Theorem 4.

Theorem 5. *(Inversion formula): If $f(x) \in L_p(0,\infty)$, $1 \le p \le 2$ [or $f(x) \in M_p(0,\infty)$, $p > 2$], $p^{-1} + q^{-1} = 1$, $\Re\left[\beta + \xi \sum_1^r N_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > -q^{-1}$; $\Re\left[\alpha + \xi \sum_1^r M_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > -p^{-1}$ for $j = 1, 2, ..., M_r$; the integrals involved are absolutely convergent and*

$$R_{\lambda_n}^{\alpha,\beta}[f(x)] = [n^*(x)], \tag{5.8}$$

then the following result holds:

$$f(x) = \int_0^\infty t^{-1}[n^*(t)][L(x/t)]\,dt, \tag{5.9}$$

where

$$[L(x)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{-s}}{\zeta[(s)]}ds \tag{5.10}$$

and

$$\zeta[(s)] = I_{\lambda_n}^{\alpha+s-1,\beta}[1]. \tag{5.11}$$

Proof. Multiplying both sides of equation (5.8) by x^{s-1}, integrating from 0 to ∞ with respect to x and applying Theorem 2, we find that

$$M[f(x)] = \frac{M[n^*(x)]}{\zeta[(s)]}.$$

Now using the inverse Mellin transform, the desired result (5.9) is obtained.

6. Some general properties of the operators

The results presented in this section can be established with the help of the definitions (2.1) and (2.2).

(6.1) $$x^{-1} I_{\lambda_n}^{\delta,\beta}\left[f\left(\frac{1}{x}\right)\right] = R_{\lambda_n}^{\delta,\beta}[f(x)],$$

(6.2) $$x^{-1} R_{\lambda_n}^{\alpha,\beta}\left[f\left(\frac{1}{x}\right)\right] = I_{\lambda_n}^{\alpha,\beta}[f(x)],$$

(6.3) $$x^{\eta} I_{\lambda_n}^{\delta,\beta}[f(x)] = I_{\lambda_n}^{\delta-\eta,\beta}[x^{\eta} f(x)],$$

(6.4) $$x^{\eta} R_{\lambda_n}^{\alpha,\beta}[f(x)] = R_{\lambda_n}^{\alpha+\eta,\beta}[x^{\eta} f(x)].$$

The following two properties express the homogeneity of the operators I and R.

If

(6.5) $$I_{\lambda_n}^{\delta,\beta}[f(x)] = [g(x)],$$

then

(6.6) $$I_{\lambda_n}^{\delta,\beta}[f(cx)] = [g(cx)].$$

If

(6.7) $$R_{\lambda_n}^{\alpha,\beta}[f(x)] = [\Phi(x)],$$

then

(6.8) $$R_{\lambda_n}^{\alpha,\beta}[f(cx)] = [\Phi(cx)].$$

We will discuss compositions of these operators in a future communication.

References

[1] P. K. Banerji, P. L. Sethi. Operators of a generalized function of n-variables. *Math. Student* **46**, 1978, 152-159.

[2] A. Erdélyi. On some functional transformations, *Univ. Politec. Torino, Rend. Sem. Mat.* **10**, 1950-51, 217-234.

[3] S. P. Goyal, R. M. Jain, Neelima Gour. Fractional integral operators involving a product of generalized hypergeometric functions and a general class of polynomials. *Indian J. Pure Appl. Math.* **22** (5), 1991, 401-411.

[4] S. L. Kalla. On operators of fractional integration. *Math. Notae*, **22**, 1970–71, 89–93.

[5] S. L. Kalla. On operators of fractional integration, II. *Math. Notae*, **25**, 1976, 29 -35.

[6] S. L. Kalla. R. K. Saxena. Integral operators involving hypergeometric functions. *Math. Z.* **108**, 1969, 231-234.

[7] S. L. Kalla. V. S. Kiryakova. A generalized fractional calculus dealing with H-functions. -In: Fractional Calculus and Applications (Proc. Int. Conf. held in Tokyo, 1989), Koriyama Press, 1990, 62-69.

[8] S. L. Kalla. V. S. Kiryakova. An H-function generalized fractional calculus based upon compositions of Erdélyi-Kober operators in L_p. *Math Japon.*, **35**, No.6, 1990, 1151-1171.

[9] V. S. Kiryakova. On operators of fractional integration involving Meijer's G-function. *Comp. Rend. Acad. Bulg. Sci.*, **39**, 1986, No.10, 25-28.

[10] H. Kober. On fractional integrals and derivatives. *Quart. J. Math.* **1**, 1940, 193-211.

[11] E. R. Love. Some integral equations involving hypergeometric functions. *J. Edinburgh Math. Soc.* (3) **15**, 1967, 169-198.

[12] M. Saigo. A remark on integral operators involving the Gauss hypergeometric functions, *Rep. College General Ed. Kyushu Univ.* **11** 1978, 135-143.

[13] M. Saigo, R. K. Saxena, J. Ram. Certain properties of operators of fractional integrations associated with Mellin and Laplace transformations. -In: Current Topics in Analytic Function Theory (Ed. H.M.Srivastava and S.Owa), World Sci. Publ., Singapore (to appear).

[14] M. Saigo, R. K. Saxena, J. Ram. On the fractional calculus operator associated with the H-function. *Ganita Sandesh*, **6**, No.1, 1992, 36-47.

[15] R. K. Saxena. On fractional integration operators. *Math. Z.* **96**, 1967, 288-291.

[16] R. K. Saxena. R. K. Kumbhat. Integral operators involving H-function. *Indian J. of Pure Appl. Math.* , **5**, 1974, 1-6.

[17] R. K. Saxena. R. K. Kumbhat. Some properties of generalized Kober operators. *Vijnana Parishad Anusandhan Patrika*, **18**, 1975, 139-150.

[18] R. K. Saxena. R. K. Kumbhat. Fractional integration operators of two variables. *Proc. Indian Acad. Sci.* **78**, 1973, 177 - 186.

[19] R. K. Saxena. G. C. Modi. Multidimensional fractional integration operators associated with hypergeometric function. *Nat. Acad. Sci. Letters* **3**, 1980, 153-157.

[20] R. K. Saxena. G. C. Modi. Multidimensional fractional integration operators associated with hypergeometric function. *Vijnana Parishad Anusandhan Patrika*, **28**, 1985, 87-97.

[21] R. K. Saxena. V. S. Kiryakova. On relation between the two-dimensional H-transforms in terms of Erdeĺyi-Kobel operators. *Math. Balkanica*, **6**, 1992, 133-140.

[22] H. M. Srivastava. A Contour integral involving Fox's H-function. *Indian J. Math.* **14**, 1972, 1-6.

[23] H. M. Srivastava, R. Panda. Some bilateral generating functions for a class of generalized hypergeometric polynomials. *J. Reine Angew. Math.* **283/284**, 1976, 265-274.

[24] H. M. Srivastava, K. C. Gupta, S. P. Goyal. The H-functions of One and Two Variables with applications, South Asian Publishers, New Delhi, 1982.

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