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On a Boundary Value Problem for an Equation of Mixed Type

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In this paper a boundary value problem for an equation of mixed type is considered and investigated. Existence of a weak solution is proved.

Let G be a bounded domain in the plane (x,y) with a partially smooth boundary Γ and let Ox intersect this domain. Consider the equation:

$$Lu \equiv u_{xx} + k(y)u_{yy} + \alpha_1(x,y)u_x + \alpha_2(x,y)u_y + \gamma(x,y)u = f(x,y), \tag{1}$$

where $k(y) \in C^2(\bar{G})$, $yk(y) > 0$ for $y \neq 0$, $\alpha_1(x,y)$ and $\alpha_2(x,y) \in C^1(\bar{G})$ and $\gamma(x,y) \in C(\bar{G})$.

For $y > 0$ equation (1) is elliptic, for $y = 0$ it is parabolic and for $y < 0$ - hyperbolic.

To formulate a correctly posed boundary value problem for (1), we reduce equation (1) to a positively symmetric system in the sense of Friedrichs [1]. To this end, we remember some facts of the theory of positively symmetric systems [1].

In the bounded domain $G \subset R^n$ with partially smooth boundary Γ, we consider the system:

$$K\hat{u} = \sum_{i=1}^{n} A^i \frac{\partial \hat{u}}{\partial x_i} + B\hat{u} = \hat{F}, \hat{u} = (u_1, u_2, ..., u_n), \tag{2}$$

$$\hat{F} = (f_1, f_2, ..., f_n),$$

where A^i and B are $n \times n$ matrices, the elements of A^i are partially smooth functions and those of B - partially continuous functions. We introduce also the matrix:

$$\varkappa = B - 0.5 \left(\sum_{i=1}^{n} \frac{\partial A^i}{\partial x_i} \right)$$

If the matrices A^i are symmetric and the matrix $\varkappa + \varkappa^*$ is positively defined in $\bar{G}$, then system (2) is said to be positively symmetric (here, $\varkappa^*$ is the transposed matrix of $\varkappa$).

We consider the boundary matrix

$$\beta = \sum_{i=1}^{n} n_i A^i(x),$$

where $n(x) = (n_1(x), n_2(x), ..., n_n(x))$ is the unit vector of outer normal towards Γ. If the matrix β is representable in the form $\beta = \beta_+ + \beta_-$, with

a) $\mu + \mu^* \geq 0$, where $\mu = (\beta_+ + \beta_-)/2$;

b) $\ker \beta_+ + \ker \beta_- = R^n$, then the boundary value condition $\beta_- u = 0$ is called admissible and $\beta_+ \mu = 0$ is a conjugate boundary value condition.

1. Reduction of equation (1) to a positively symmetric system of first order and investigation of the system obtained

Let $u(x, y) \in C^2(\bar{C})$ and satisfy equation (1) in G. Then the functions $u_o = u$, $u_1 = u_x$, $u_2 = u_y$ satisfy in G the following system:

$$(3) \quad \left| \begin{array}{l} u_2 - \dfrac{\partial u_0}{\partial y} = 0 \\ \dfrac{\partial u_1}{\partial x} + k(y)\dfrac{\partial u_2}{\partial y} + \alpha_1(x,y)u_1 + \alpha_2(x,y)u_2 + \gamma(x,y)u_0 = f(x,y) \\ -\dfrac{\partial u_1}{\partial y} + \dfrac{\partial u_2}{\partial x} = 0. \end{array} \right.$$

Let us rewrite system (3) in a matrix form and multiply it on the left-hand side by a suitably chosen matrix:

$$Z = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & b & 0 \end{bmatrix}, \qquad \det Z = -ab^2,$$

where $a(x,y)$ and $b(x,y)$ are smooth functions (to be defined later). Then we obtain the symmetric system:

$$(4)\qquad \hat{L}\hat{u} \equiv \hat{A}^1\frac{\partial \hat{u}}{\partial x} + \hat{A}^2\frac{\partial \hat{u}}{\partial y} + \hat{B}\hat{u} = \hat{f}, \quad \hat{u} = (u_0, u_1, u_2), \quad \hat{f} = (0,0,bf),$$

and

$$\hat{A}^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & b & 0 \end{bmatrix}, \quad \hat{A}^2 = \begin{bmatrix} -a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & kb \end{bmatrix} \quad \text{and} \quad \hat{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma b & b\alpha_1 & b\alpha_2 \end{bmatrix}.$$

For the matrix $\varkappa + \varkappa^*$ to be positively defined in $\bar{G}$, it is necessary and sufficient that: 1) $a_y > 0$, 2) $a_y b_y > 0$, 3) $\det \varkappa + \varkappa^*) > 0$. We choose the function $b = b_1 + b_2 y$, where the coefficients satisfy the following conditions:

a) b_2 is a fixed arbitrary positive number;

b) b_1 satisfies conditions $b_1 + b_2 y > 0$ for $y \in [h_1, h_2]$ ($h_1 = \min_{\bar{G}} y$, $h_2 = \max_{\bar{G}} y$), and $(2\alpha_2 - k')b_1 - kb_2 \geq 2M = const > 0$ in $\bar{G}$, simultaneously, with the additional requirement $2\alpha_2 - k' \geq \tau = const > 0$ in $\bar{G}$.

We choose $a = -1/(\tau y + \delta)$, where $\delta = -1 - \tau h_2$.

The function $\alpha_1(x,y)$ is chosen sufficiently small in absolute value so that $b_2[(2\alpha_2 - k')b_1 - kb_2] - b^2\alpha_1^2 \geq M > 0$ and $\gamma(x,y)$ is such that $\det(\varkappa + \varkappa^*) > Ma^2\tau/3 > 0$ in $\bar{G}$.

For $a(x,y)$ and $b(x,y)$ chosen in this way and the above additional propositions for the coefficients $\alpha_1(x,y)$, $\alpha_2(x,y)$ and $\gamma(x,y)$, it is easy to check that $\det Z \neq 0$ and the matrix $\varkappa + \varkappa^*$ is positively defined. Therefore, system (3) and (4) are equivalent in G and system (4) is positively symmetric.

We introduce the boundary matrix

$$(5)\qquad \beta = \begin{bmatrix} -an_y & 0 & 0 \\ 0 & -bn_y & bn_x \\ 0 & bn_x & kbn_y \end{bmatrix},$$

where (n_x, n_y) is the unit vector of the outer normal towards Γ. We consider the quadratic form:

$$\hat{u} \cdot \beta\hat{u} = -an_y u_0^2 - bn_y u_1^2 + kbn_y u_2^2 + 2bn_x u_1 u_2$$

For $bn_y \neq 0$ this form can be written in the form:

$$(6)\qquad \hat{u} \cdot \beta\hat{u} = -an_y u_0^2 + b[(n_x^2 + kn_y^2)u_2^2 - (u_1 n_y - u_2 n_x)^2]/n_y.$$

It is seen from here how some boundary value problems can be stated for system (3) so that the boundary conditions to be admissible.

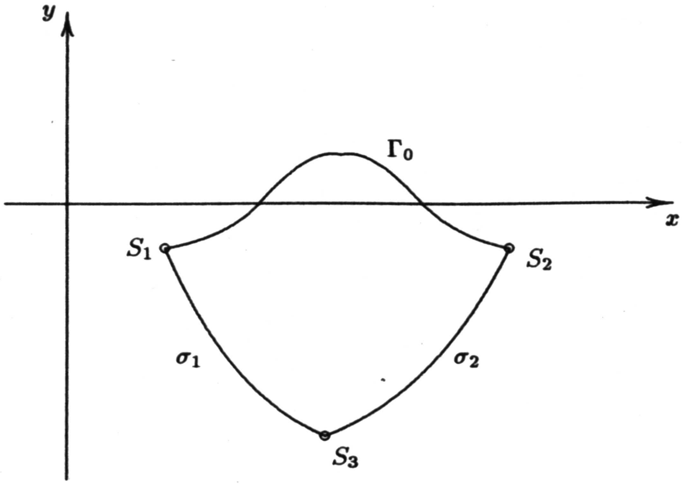

Figure 1

Let the boundary of the domain be $\Gamma = \Gamma_0 \cup \sigma_1 \cup \sigma_2$ (Fig.1), where σ_1 and σ_2 are characteristics of equation (1) and Γ_0 is two times smooth curve. On σ_1 and σ_2 we have $n_x^2 + kn_y^2 = 0$ and $n_y < 0$, hence it follows that $\hat{u} \cdot \beta\hat{u} > 0$. If we choose $\beta_- = 0$, $\beta_+ = \beta$, then the conditions of Friedrichs [1] are satisfied and therefore σ_1 and σ_2 are freed from boundary values.

We assume that on Γ_0 the following additional conditions are fulfilled:

$$n_y > 0, \quad kn_y^2 + n_x^2 > 0 \tag{7}$$

Then, we define following quadratic form:

$$\hat{u} \cdot 2\mu\hat{u} = an_y u_0^2 + b[(n_x^2 + kn_y^2)u_2^2 + (u_1 n_y - u_2 n_x)^2]/n_y \tag{8}$$

The symmetric matrix:

$$2\mu = \begin{bmatrix} an_y & 0 & 0 \\ 0 & bn_y & -bn_x \\ 0 & bn_x & (2n_x^2 + kn_y^2)b/n_y \end{bmatrix},$$

defined by (8) is positively defined on Γ_0. We solve the system:

$$\left| \begin{aligned} 2\mu &= \beta_+ - \beta_- \\ \beta &= \beta_+ + \beta_- \end{aligned} \right.$$

For this choice of β_+ and β_-, the condition $\beta_- u = 0$ is admissible. On Γ_0 and upon conditions (7), $\beta_- u = 0$ has the form:

(9) $$u_0 = 0 \quad \text{and} \quad u_1 n_y - u_2 n_x = 0 \text{ on } \Gamma_0.$$

Then the well-posed boundary value problem for system (4) is:

Problem A'. *Find a solution of system (4) satisfying boundary value conditions (9).*

Denote by $\hat{L}_2(G)$ the Hilbert space with the scalar multiplication:

$$(\hat{u}, \hat{v}) = \int_G (u_0 v_0 + u_1 v_1 + u_2 v_2) dx dy,$$

where

$$\hat{u} = (u_0, u_1, u_2) \quad \text{and} \quad \hat{v} = (v_0, v_1, v_2).$$

The norm in $\hat{L}_2(G)$ is denoted by $\|\cdot\|_{\hat{L}_2(G)}$

Definition. *The function $\hat{u} \in \hat{L}_2(G)$ is said to be a weak solution of a problem A', if $(L^*\hat{v}, \hat{u}) = (\hat{v}, \hat{f})$ ($\hat{L}^*$ is a formally conjugate of $\hat{L}$ operator) for each partially smooth function in G, satisfying the conjugate boundary value conditions:*

(10) $$a v_1 + b v_2 = 0 \text{ on } \Gamma_0 \text{ and}$$

$$v_0 = 0, \quad v_1 n_y - v_2 n_x = 0 \text{ on } \sigma_1 \cup \sigma_2$$

Definition. *The function $\hat{u} \in \hat{L}_2(G)$ is said to be a strong solution of problem A', if it exists a sequence of partially smooth in G functions $\hat{u}^k$ satisfying boundary value conditions (9) and:*

$$\|\hat{u}^k - \hat{u}\|_{\hat{L}_2(G)} \longrightarrow 0, \quad \|\hat{L}\hat{u}^k - \hat{f}\|_{\hat{L}_2(G)} \longrightarrow 0, \quad k \to \infty$$

From the Friedrichs theory [1,2] for positively symmetric systems it follows that the energy inequalities are fulfilled, namely:

(11) $$\|\hat{L}\hat{u}\|_{\hat{L}_2(G)} > C\|\hat{u}\|_{\hat{L}_2(G)}, \quad C > 0, \quad C = const,$$

for all the partially smooth in $\bar{G}$ functions $\hat{u}$ satisfying boundary value conditions (9) and:

$$(12) \qquad \|\hat{L}^*\hat{v}\|_{\hat{L}_2(G)} > C_0\|\hat{v}\|_{\hat{L}_2(G)}, \quad C_0 > 0, \quad C_0 = const,$$

for all the partially smooth in G functions v satisfying the conjugate boundary value conditions (10).

It follows immediately from (11), that if problem A' has a strong solution it is unique.

As it is known from [3] it follows from (12), that problem A' has a weak solution for each function $\hat{f} \in \hat{L}_2(G)$. Then, we have proved that the following theorem holds:

Theorem 1. *Problem A' has a weak solution and if the strong solution exists, it is unique.*

We are to prove the following:

Theorem 2. *Each weak solution of problem A' is a strong solution too.*

Proof. For the inner points of G the result follows immediately from [1,2]. Let us consider again the boundary matrix β (5), $\det\beta = ab^2n_y(kn_y^2+n_x^2)$. The boundary Γ is two times smooth everywhere except for the points S_1, S_2 and S_3 (Fig.1). On Γ_0, the rank of the matrix β is maximal and fixed, since $\det\beta \neq 0$. The boundary space $N(x) = \{(0, u_1, u_2); u_1n_x - n_yu_2 = 0\}$ is smooth. Therefore in a small neighborhood of an arbitrary point of Γ_0, all the conditions for coincidence of the weak and strong solution in the theorems of Lax and Phillips [4] are fulfilled. On the characteristics σ_1 and σ_2, $\det\beta = 0$, but the principle minor of second order

$$\begin{bmatrix} an_y & 0 \\ 0 & -bn_y \end{bmatrix} = abn_y^2 \neq 0.$$

The rank keeps on to be 2 in a neighbourhood of characteristics too. Then, from [5] it follows that the weak solutions is also strong one. The points S_1, S_2 and S_3 are corner points, vertices of angles different from zero and smaller then π (with respect to the domain). On one of the sides of the angle, the boundary matrix keeps a constant rank, and on the other side only boundary value conditions or only conjugate boundary value conditions are given. Therefore, the results of Lax and Phillips [4], Peyser [5], Popivanov [6] for coincidence of the weak and the strong solution are applicable. This completes the proof.

Theorem 1 and Theorem 2 yield the following theorem.

Theorem 3. *Problem A' has an unique strong solution.*

2. Weak solvability of the corresponding boundary value problem for equation (1)

Problem A. *Find a solution of equation (1) in domain G (under the above assumptions for the boundary Γ) satisfying the boundary value conditions:*

$$u(x,y) = 0 \text{ on } \Gamma_0 \tag{13}$$

$$u(x,y)\,|_{\sigma_1 \cup \sigma_2} \sim$$

(the sign $\sim$ denotes that no boundary value conditions are prescribed).

The following assumptions and requirements for the coefficients of equation (1) hold, and ensuring that system (3) is positively symmetric.

It is easily seen that the conjugate boundary value conditions for equation (1) are:

$$v = 0 \text{ on } \Gamma_0 \cup \sigma_1 \cup \sigma_2 \tag{14}$$

Denote by $W_2^2(\bar{G})$ and $W_{2,*}^2(\bar{G})$ the closure with respect to the norm $W_2^2(G)$ of the set of functions of $C^2(\bar{G})$, satisfying (13) and (14).

Definition. *The function $u \in L_2(G)$ is said to be a weak solution of problem A, if $(u, L_v^*) = (f, v)$ for each $v \in W_{2,*}^2(G)$.*

Theorem 4. *The boundary value problem A has a weak solution $u \in L_2(G)$ for each $f \in L_2(G)$.*

Proof. Let $\hat{u} = (u_0, u_1, u_2)$ is the strong solution of problem A'. Then there exists a sequence $\hat{u}^k = (u_0^k, u_1^k, u_2^k)$ of functions smooth in G and satisfying boundary value conditions (9), for which:

$$\|\hat{u}^k - \hat{u}\| \to 0, \quad \|L\hat{u}^k - \hat{f}\| \to 0 \quad ,k \to \infty$$

hold.

We assume that G is defined by the inequalities: $\{x_1 < x < x_2; \quad y_-(x) \leq y \leq y_+(x)\}$. Then for $(x, y) \in G$ we define the function [7]:

$$\theta^k(x, y) = \int_{y_+(x)}^{y} u_2^k(x, t)dt; \quad \theta^k(x, y) \in C^1(G).$$

From the fact that the sequence $\{\hat{u}^k\}$, is fundamental and therefore, the sequence $\{u_2^k\}$ is also fundamental it follows that it is the same for the sequence $\{\theta^k(x, y)\}$. The completeness of the space $L_2(G)$ yields that a function $\theta(x, y) \in L_2(G)$ exists, so that:

$$\|\theta^k x, y) - \theta(x, y)\|_{L_2(G)} \to 0 \text{ as } k \to \infty \tag{15}$$

It is easy to prove that

$$\|\theta^k x, y) - u_0(x, y)\|_{L_2(G)} \to 0 \text{ as } k \to \infty \tag{16}$$

Therefore from (15) and (16) it follows that $u_0(x, y) = \theta(x, y)$.

We prove that

$$\left\| \frac{\partial \theta^k}{\partial x} - u_1 \right\|_{L_2(G)} \to 0 \tag{17}$$

and

$$\left\| \frac{\partial \theta^k}{\partial y} - u_2 \right\|_{L_2(G)} \to 0 \text{ as } k \to \infty.$$

From (16) and (17) it follows that

$$u_1 = \frac{\partial u_0}{\partial x}, \quad u_2 = \frac{\partial u_0}{\partial y}$$

and $u_0(x, y) \in W_2^1(G) \subset L_2(G)$.

By the help of an imbedding theorem of Sobolev [8], it is shown that

$$\|u_0\|_{L_2(\Gamma_0)} = 0,$$

i.e. $u_0 = 0$ almost everywhere on Γ_0. It is verified that $u_0(x, y)$ is a weak solution of problem A. This proves the theorem.

Let $u \in C^2(\bar{G})$ and $u = 0$ on Γ_0. Then $\hat{u} = (u, u_x, u_y)$ satisfies system (4) and boundary value condition (9). Then the estimation (11) holds:

$$\|\hat{L}\hat{u}\|_{L_2(G)} \geq C_1\|\hat{u}\|_{L_2(G)},$$

i.e.

$$\left[\int_G b^2(Lu)^2 dxdy\right]^{1/2} \geq C\left[\int_G [u^2 + u_x^2 + u_y^2]dxdy\right]^{1/2}$$

Since the function $b(x,y)$ is bounded in $\bar{G}$, then

$$\|Lu\|_{L_2(G)} \geq C_1\|u\|_{W_2^1(G)}. \tag{18}$$

It is seen that the estimation (18) holds for every function $u \in W_2^1(G)$, which satisfies equation (1) and boundary condition (13).

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