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Exponentially Fitted Petrov–Galerkin Method of Finite Elements

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We give an exponentially fitted polynomials Petrov-Galerkin method of finite elements, applied to singular self-adjoint problem. We derive an exponentially fitted finite difference scheme and we give an error bounds. We confirm it numerically.

1. Introduction.

Polynomials Petrov-Galerkin method of finite elements has no ability to follow the exponential feature of the exact solution expecially in boundary layer when the solution changes very rapidly [4]. But the exponential functions suit to this kind of problems and because of that we fit trial space [3], [5] with piecewise exponentials. The difference scheme derived in this way has a second order of uniform convergence in a small parameter ϵ..

We recall that the scheme is uniform in a small parameter ϵ and it is of order p (see [2]) iff

$$|u(x_i) - u_i| \leq h^p$$

where $u(x_i)$ is the exact solution at the point x_i and u_i is the computed one, h is the mesh size, M is a constant independent of mesh size h and perturbation parameter ϵ.

In the second Section of the paper we give a description of Petrov-Galerkin polynomial method of finite elements. We apply it to the singularly perturbed two-point boundary value self-adjoint problem. Therefore, in the third Section we generate the difference scheme and we determine parameter which give an exponential means to Galerkin method. We give an error estimate in uniform norm concerning small parameter ϵ. In the Section 4. we confirm experimentally the theoretical predictions.

Note. We suppose uniform partition of interval $[0,1]$ with $h = 1/n, x_i = ih, i = 0(1)n-1, n$ being an integer. Throughout the paper M will denote different constants independent of h and ϵ. R will denote part in error estimate which is negligible. We denote the truncation error of difference scheme by $\tau_i(u)$. We have $\tau_i(u) = Ru_i - QLu_i$, where R and L are corresponding operators. By $\bar{\rho}$ we denote $h\epsilon^{-1/2}$. Then $\rho_i = \rho_i^{1/2}\bar{\rho}$, where $p_i = p(x_i)$.

2. Petrov-Galerkin method of finite elements.

Consider two-point singularly perturbed self-adjoint problem

$$Lu = -\epsilon u'' + p(x)u = f(x), \quad u(0) = A, u(1) = B \tag{1}$$

where ϵ is a small parameter, $0 < \epsilon \ll 1$, functions $p(x), f(x)$ are sufficiently smooth and $p(x)$ satisfies condition $p(x) \geq \bar{p} > 0$.

From [4] $u \in H_0^1$ is the Galerkin (or weak) solution of (1) iff $u \in H_0^1(0,1)$ and

$$B_\epsilon(u,v) = \epsilon(u',v') + p(u,v) = (f,v),$$

for all v from $H_0^1(0,1)$, where $(\cdot,\cdot)$ denotes innerproduct in $L^2(0,1)$.

We choose two spaces of finite equal dimensions T^h and V^h called as test and trial spaces. Let $\{\phi_i\}, i = 1(1)n$ and $\{\psi_i\}, i = 1(1)n$ be the basis of the trial and test space respectively. Then the Galerkin approximation is:

Find an $\{u_i\}$ so that

$$u_\epsilon = \sum_{i=1}^{n} u_i\phi_i, \quad u_0 = A, \quad u_n = B$$

where $\{u_i\}, i = 0(1)n$ satisfy system of equations

$$\sum_{k=i-1}^{k=i+1} B_\epsilon(\phi_k, \psi_i) = (f, \psi_i), \quad i = 1(1)n-1.$$

We use polynomials spaces of Christie et.al.[1] which uses linear (hat) functions

$$\phi_i(x) = \phi(x/h - i), \quad i = 1(1)n,$$

as a trial spaces, where

$$\phi(s) = \begin{cases} 0 & |s| > 1 \\ 1 + s - 1 \leq & s < 0 \\ 1 - s0 \leq & s \leq 1, \end{cases}$$

or for the test space we use

$$\psi_i(x) = \phi_i(x) + \alpha(x)\delta(x/h - i), \quad i = 1(1)n$$

where

$$\delta(s) = \begin{cases} 0 & |s| > 1 \\ -3s(s-1)0 \leq & s < 1 \\ -\delta(-s) - 1 \leq & s \leq 0. \end{cases}$$

and $\alpha(x)$ is exponential function which we determine later.

The properties of functions which belong to these spaces are:

(i)supp$(\phi_i(x)) = [x_{i-1}, x_{i+1}]$; $(ii)\phi_i(x_i) = 1$; $(iii) \sum_{i=1}^{n} \phi_i(x) = 1$, for all x from $[x_1, x_{n-1}]$.

Our bilinear form is now:

$$B_\epsilon(\phi_i, \psi_i) = \epsilon(\phi_i', \psi_i') + \bar{p}_i(\phi_i, \psi_i) = (\bar{f}_i, \psi_i), i = 0(1)n.$$

For $\bar{p}_i$ and $\bar{f}_i$ we use a piecewise constants of the form $\bar{p}_i = (p_i + p_{i-1})/2$; $\bar{f}_i$ also. From this method we obtain the difference scheme:

$$(2) \qquad Ru_i = Qf_i(h^2/\epsilon), \quad u_0 = A, \ u_1 = B, i = 1(1)n - 1$$

or

$$r_i^- u_{i-1} + r_i^c u_i + r_i^+ u_{i+1} = (q_i^- f_{i-1} + q_i^c f_i + q_i^+ f_{i+1})(h^2/\epsilon),$$

where the coefficients of the scheme are given by:

$$r_i^- = -1 + \rho_i^2(1/6 + \alpha_i/4), \quad r_i^+ = -1 + \rho_{i+1}^2(1/6 - \alpha_{i+1}/4),$$

$$r_i^c = 2 + \rho_i^2(1/3 + \alpha_i/4) + \rho_{i+1}^2(1/3 - \alpha_{i+1}/4),$$

$$q_i^- = 1/4(1 + \alpha_i), \quad q_i^+ = 1/4(1 - \alpha_{i+1}), \quad q_i^c = q_i^- + q_i^+,$$

$$\rho_i = p_i\bar{\rho}^2, \quad \bar{\rho} = h\epsilon^{-1/2}, p_i = p(x_i).$$

We determine α_i so that the truncation error of the scheme (2) would be equal to zero, or $\tau_i = Ru_i - QLu_i = 0$, where

$$Ru_i = \bar{u}_{0i}\{r_i^- exp(\rho_0) + r_i^c + r_i^+ exp(-\rho_0)\}$$

$$QLu_i = (h^2/\epsilon)\bar{u}_{0i}\{(p_0 - p_{i-1})q_i^- exp(\rho_0) + (p_0 - p_i)q_i^c + (p_0 - p_{i+1})q_i^+ exp(-\rho_0)\}, \bar{u}_{0i} = exp(-(p_0/\epsilon)^{1/2}x_i).$$

From $Ru_i = 0$ since $QLu_i = 0$ for $p = const$ we obtain

$$\alpha_i = 2/(\rho_i^2 \sinh \rho_i)[4 \sinh^2 \rho_i/2 - \rho_i^2/3 \cosh \rho_i - 2/3\rho_i^2]$$

where $\rho_i = \bar{\rho}p_i^{1/2}$, $p_i = p(x_i)$, $\rho_0 = \bar{\rho}p_0^{1/2}$. In order to obtain more simmetric fitting factor we set $p_{i-1} = p_{i+1} = p_i$ and we obtain

$$\alpha_{i1} = 1/((\rho_i^2 - \rho_0^2/2) \sinh \rho_0)\{4 \sinh^2 \rho_0/2 - 1/3\rho_i^2 \cosh \rho_0 - 2/3\rho_i^2 + (\rho_i^2 - \rho_0^2)/2 \cosh \rho_0\}.$$

3. Proof of the uniform convergence.

We shall prove a second order of uniform accuracy for the scheme (2). We have

$$|u(x_i) - u_i| \leq M\ ||A||^{-1} \max_i |\tau_i(u)| \tag{1}$$

where $\tau_i(u)$ is the truncation error of the scheme (2), and A is the matrix of lynear system of equation (2). Since $||A||^{-1} \geq (r_i^- + r_i^c + r_i^+)^{-1} \geq M(\rho_i^2)^{-1}$, where $||.||$ is the usual max norm, we obtain by simple calculation the matrix estimate of the discretization (2):

$$||A||^{-1} \geq M/\bar{\rho}^2. \tag{2}$$

In the estimate of truncation error we shall use the asymptotical expansion of the exact solution given by Doolan et.al.[2].

Lemma 1. ([2]) *Let $u(x) \in C^4[0,1]$ and $p'(0) = p'(1) = 0$ then the solution of (1) can be expressed as*

$$u(x) = u_0(x) + w_0(x) + g(x), \quad where \tag{3}$$

$$u_0(x) = p_0 exp[-x(p(0)/\epsilon)^{1/2}]$$
$$w_0(x) = p_1 exp[-(1-x)(p(1)/\epsilon)^{1/2}]$$

p_0, p_1 *are bounded functions of* ϵ *independent of* x *and*

(4) $$|g^i(x)| \le M(1+\epsilon^{1-i/2}), i = 0(1)4.$$

Lemma 2. *Let* $p = const, f(x) \in C^2[0,1]$. *Denote by* $\{u_i\}, i = 1(1)n-1$ *the solution of lynear system (2). Then the following estimate holds:*

(5) $$|u(x_i) - u_i| \le Mh^2.$$

Proof. When $p = const$ we determined α_i so that truncation error $\tau_i(u_0) = 0$. Similarly, for another boundary layer function we have $\tau_i(w_0) = 0$ for $p = const$. So we must estimate only part g in (5). We have

(6) $$\tau_i(g) = \tau_i^{(0)} g^{(0)} + \tau_i^{(1)} g^{(1)} + \tau_i^{(2)} g^{(2)} + \ldots + R.$$

Since when $p = const, \tau_i^{(0)} = \tau_i^{(1)} = 0$. We have

$$\tau_i^{(2)} = h^2\{(r_i^- + r_i^+)/2 + (q_i^- + q_i^c + q_i^+) - \rho_i^2/2(q_i^+ + q_i^-)\}.$$

and $\tau_i^{(2)} = h^2(-\rho_i^2/12)$, which gives with (6) the estimate

$$|\tau_i^{(2)} g''| \le M\,\bar{\rho}^2,$$

for all $\bar{\rho}$. Then,

$$\tau_i^{(3)} = h^3\{(r_i^+ - r_i^-)/6 + (q_i^+ - q_i^-) - \rho_i(q_i^+ - q_i^-)/6\}$$

and $\tau_i^{(3)} = h^3(-\alpha_i/2)$. When $\bar{\rho} \le 1$

$$\alpha_i = -\bar{\rho}/6(1 - \bar{\rho}^2/6) + O(\bar{\rho}^4),$$

which gives

$$|\tau_i^{(3)}| \le Mh^3\bar{\rho}.$$

In the oposite case, when $\bar{\rho} \ge 1$, we have $\alpha_i \le M$. It yields with (6)

$$|\tau_i^{(3)} g^{(3)}| \le Mh^2\bar{\rho}\min(\bar{\rho}, 1).$$

The higher derivatives are of the lower order. So we obtain in (8)

(7) $$|\tau_i(g)| \le Mh^2\bar{\rho}^2.$$

From (3), matrix estimate (4) and (9) we obtain (7).

Theorem 1. *Let $p(x), f(x) \in C^2[0,1]$. Let $\{u_i\}, i = 1(1)n-1$ be the approximate solution for (1) obtain by exponentially fitted Petrov-Galerkin method described above, then the following estimate holds:*

$$|u(x_i) - u_i| \leq Mh^2. \tag{8}$$

Proof. From Lemma 2. $\tau_i^{(0)} = \tau_i^{(0)} = 0$, and $|\tau_i(g)| \leq M\ h^2\bar{\rho}^2$ when $p = const$. We have

$$\tau_i(g) = \tau_i(g)(\rho) + (\rho_{i-1} - \rho)\frac{\partial \tau_i}{\partial \rho_{i-1}}(\rho) + (\rho_{i+1} - \rho)\frac{\partial \tau_i}{\partial \rho_{i+1}} + R,$$

where $\rho = h(p_i/\epsilon)^{1/2}$.

By simple calculation we get

$$|\tau_i(g)| \leq Mh^2\bar{\rho}^2 + O(h^3\bar{\rho}\min(\bar{\rho}, 1)).$$

From (3) and (4) and previous estimate we obtain

$$|g(x_i) - g_i| \leq Mh^2. \tag{9}$$

We showed that for $p = const, \tau_i(u_0) = 0$ and because of that

$$\tau_i(u) = (\tau_i(u_0)(\rho) = 0) + (\rho_{i-1} - \rho)\frac{\partial \tau_i}{\partial \rho_{i-1}}(\rho) + (\rho_{i+1} - \rho)\frac{\partial \tau_i}{\partial \rho_{i+1}}(\rho) + R.$$

Since

$$|\rho_{\stackrel{+}{i-1}} - \rho| \leq M\ h\bar{\rho}, \quad |\rho_{i-1} - \rho + \rho_{i+1} - \rho| \leq M\ h^2\bar{\rho}$$

and

$$|\frac{\partial \tau_i}{\partial \rho_{\stackrel{+}{i-1}}}(\rho)| \leq M\ \bar{\rho}$$

we have that

$$|\tau_i(u_0)| \leq Mh^2\bar{\rho}^2,$$

for all $\bar{\rho}$. From this estimate, (3), and (4) we obtain

$$|u_0(x_i) - u_{0i}| \leq M\ h^2. \tag{10}$$

For $w_0(x)$ we obtain the same . Then (11), (12) and the estimate for w_0 with (5) conclude the outline of the proof.

4. Numerical evidence.

We present numerical test to illustrate the previous theory. The computations were performed in Fortran 5., double precision on PC-computer.

We applied the difference scheme (2) to the problem

$$(1)\qquad -\epsilon u'' + u = -\cos\pi x - 2\epsilon\pi^2\cos 2\pi x \quad u(o) = u(1) = 0,$$

with the exact solution

$$u(x) = (exp(-\frac{(1-x)}{\sqrt{\epsilon}}) + exp(-\frac{x}{\sqrt{\epsilon}}))/(1 + exp(-\frac{1}{\sqrt{\epsilon}}) - \cos 2\pi x^2.$$

(see [2]). We use the double mesh principle from([2]) to compute rates and maximum errors. Recall ([2]),

$$rate = \{ln(E^1) - ln(E^2)\}/ln2$$

where E^1 and E^2 are

$$E^{(\cdot)} = \max_{0\le i\le N} |u_i^N - u_{2i}^{2N}|$$

for the mesh lenght $h = 1/N$ and $h = 1/(2N)$, respectively. We started with $N = 8$ and ended by $N = 512$. The number of iteration is denoted by k. Results are displaied in Table 1.

k	0	1	2	3	4	5
ϵ	rate	rate	rate	rate	rate	rate
1	1.99	2.00	2.00	2.00	2.00	2.00
2^{-1}	1.99	2.00	2.00	2.00	2.00	2.00
2^{-2}	1.99	1.99	2.00	2.00	2.00	2.00
2^{-3}	1.99	2.00	2.00	2.00	2.00	2.00
2^{-4}	1.99	2.00	2.00	2.00	2.00	2.00
2^{-5}	1.96	1.97	1.99	2.00	2.00	2.00
2^{-6}	1.93	1.98	1.99	2.00	2.00	2.00
2^{-7}	1.97	1.98	1.99	2.00	2.00	2.00
2^{-8}	1.97	1.99	1.99	2.00	2.00	2.00
2^{-9}	1.94	1.99	1.99	2.00	2.00	2.00

Table 1.

We also find $MAX = \max_{0\le i\le N} |u(x_i) - u_i|$, where $u(x_i)$ is the exact solution of (13) and u_i is the approximate one attained at mesh points for special N. See Table 2.

ϵ/N	16	32	64	128	256	512
2^{-5}	.27E-02	.51E-03	.20E-03	.35E-04	.73E-05	.26E-05
2^{-9}	.27E-2	.51E-03	.20E-03	.35E-04	.73E-05	.26E-05
10^{-4}	.88E-03	.40E-03	.12E-03	.35E-04	.72E-05	.26E-05
10^{-6}	.39E-03	.24E-03	.67E-04	.30E-04	.62E-05	.26E-05

Table 2. MAX

In a case of variable parameter $p(x)$ we consider the following problem

$$-\epsilon u'' + (1+x)^2 u = 4(3x^2 - 3x + 2)(1+x)^2$$

$$u(0) = -1, u(1) = 0,$$

taken from [2]. The rates of uniform convergence for this problem are displaied in Table 3. p_ϵ is the everage rate for the same ϵ.

k	0	1	2	3	4	p_ϵ
1	1.91	1.98	2.00	2.00	2.00	1.98
10^{-1}	1.93	1.98	2.00	2.00	2.00	1.98
10^{-2}	2.10	2.03	2.01	2.00	2.00	2.03
10^{-3}	2.55	2.33	2.11	2.03	2.01	2.21
10^{-4}	2.09	2.52	2.54	2.25	2.08	2.30
10^{-5}	1.93	2.01	2.28	2.60	2.44	2.25

Table 3.

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