Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New series Vol.8, 1994, Fasc.2–3

The Non-Commutative Neutrix Product of the Distributions x_+^{-r} and x_-^{-s}

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The non-commutative neutrix product of the distributions x_+^{-r} and x_-^{-s} is evaluated for $r, s = 1, 2, \dots$. Further products are then deduced.

In the following, we let N be the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \dots, n, \dots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \qquad \lambda > 0, \quad r = 1, 2, \dots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

We now let $\rho(x)$ be any infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t)\rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.
The following definition for the neutrix product of two distributions was given in [4].

Definition 1.. Let f and g be distributions in $\mathcal{D}'$ and let $g_n(x) = (g * \delta_n)(x)$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to the distribution h on the interval (a, b) if

$$\underset{n\to\infty}{\text{N}-\lim}\langle f(x)g_n(x), \phi(x)\rangle = \langle h(x), \phi(x)\rangle$$

for all functions ϕ in $\mathcal{D}$ with support contained in the interval (a, b).
Note that if

$$\lim_{n\to\infty}\langle f(x)g_n(x), \phi(x)\rangle = \langle h(x), \phi(x)\rangle,$$

we simply say that the *product* $f.g$ exists and equals h, see [4]. It is obvious that if the product $f.g$ exists then the neutrix product $f \circ g$ exists and $f.g = f \circ g$. The following theorem holds, see [4].

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix products $f \circ g$ and $f \circ g'$ (or $f' \circ g$) exist on the interval (a, b). Then the neutrix product $f' \circ g$ (or $f \circ g'$) exists on the interval (a, b) and*

$$(f \circ g)' = f' \circ g + f \circ g'$$

on the interval (a, b).

The next theorem was proved in [5].

Theorem 2. *The neutrix products $\ln x_+ \circ \delta^{(r)}(x)$ and $\delta^{(r)}(x) \circ \ln x_+$ exist and*

$$\begin{aligned}\ln x_+ \circ \delta^{(r)}(x) =& [c(\rho) + \tfrac{1}{2}\psi(r)]\delta^{(r)}(x),\\ \delta^{(r)}(x) \circ \ln x_- =& c(\rho)\delta^{(r)}(x)\end{aligned}$$

for $r = 0, 1, 2, \dots$, *where*

$$c(\rho) = \int_0^1 \ln t\rho(t)\, dt$$

and

$$\psi(r) = \begin{cases} cc0, & r = 0, \\ \sum_{i=1}^{r} i^{-1}, & r \geq 1. \end{cases}$$

It was shown in [5] that by suitable choice of the function ρ, $c(\rho)$ can take any negative value.

In the following, the distributions $F(x_+, -r)$, $F(x_-, -r)$, x_+^{-r} and x_-^{-r} are defined by

$$\langle F(x_+, -r), \phi(x)\rangle = \int_0^\infty x^{-r}\Big[\phi(x) - \sum_{i=0}^{r-2} \frac{x^i}{i!}\phi^{(i)}(0) + \tag{1}$$
$$- \frac{x^{r-1}}{(r-1)!}\phi^{(r-1)}(0)H(1-x)\Big]\, dx,$$

$$\langle F(x_-, -r), \phi(x)\rangle = \int_0^\infty x^{-r}\Big[\phi(-x) - \sum_{i=0}^{r-2} \frac{(-x)^i}{i!}\phi^{(i)}(0) + \tag{2}$$
$$- \frac{(-x)^{r-1}}{(r-1)!}\phi^{(r-1)}(0)H(1-x)\Big]\, dx,$$

$$x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!}(\ln x_+)^{(r)}, \quad x_-^{-r} = -\frac{1}{(r-1)!}(\ln x_-)^{(r)},$$

for $r = 1, 2, \dots$, where H denotes Heaviside's function.

Note that the distributions $F(x_+, -r)$ and $F(x_-, -r)$ which we have just defined, were used by Gel'fand and Shilov [8] to denote the distributions x_+^{-r} and x_-^{-r} respectively.

It was proved in [3] that

$$x_+^{-r} = F(x_+, -r) + \frac{(-1)^r \psi(r-1)}{(r-1)!}\delta^{(r-1)}(x), \tag{3}$$

$$x_-^{-r} = F(x_-, -r) - \frac{\psi(r-1)}{(r-1)!}\delta^{(r-1)}(x), \tag{4}$$

for $r = 1, 2, \dots$.

The next two theorems were proved in [6] and [7] respectively.

Theorem 3. *The neutrix product* $\delta^{(r)}(x) \circ \delta^{(s)}(x)$ *and*

$$\delta^{(r)}(x) \circ \delta^{(s)}(x) = 0, \tag{5}$$

for $r, s = 0, 1, 2, \dots$.

Theorem 4. *The neutrix products* $x_+^{-s} \circ \delta^{(r)}(x)$, $\delta^{(r)}(x) \circ x_+^{-s}$, $x_-^{-s} \circ \delta^{(r)}(x)$ *and* $\delta^{(r)}(x) \circ x_-^{-s}$ *exist and*

$$x_+^{-s} \circ \delta^{(r)}(x) = \frac{(-1)^s r!}{2(r+s)!}\delta^{(r+s)}(x), \tag{6}$$

$$\delta^{(r)}(x) \circ x_+^{-s} = 0, \tag{7}$$

$$x_-^{-s} \circ \delta^{(r)}(x) = \frac{r!}{2(r+s)!}\delta^{(r+s)}(x), \tag{8}$$

$$\delta^{(r)}(x) \circ x_-^{-s} = 0, \tag{9}$$

for $r = 0, 1, 2, \dots$ *and* $s = 1, 2, \dots$.

We now prove the following theorem.

Theorem 5. *The neutrix products* $F(x_+, -r) \circ x_-^{-s}$, $x_+^{-r} \circ F(x_-, -s)$, $x_+^{-r} \circ x_-^{-s}$ *and* $F(x_+, -r) \circ F(x_-, -s)$ *exist and*

$$F(x_+, -r) \circ x_-^{-s} = \frac{(-1)^r c(\rho)}{(r+s-1)!}\delta^{(r+s-1)}(x), \tag{10}$$

$$x_+^{-r} \circ F(x_-, -s) = (-1)^r \frac{c(\rho) + \frac{1}{2}\psi(s-1)}{(r+s-1)!}\delta^{(r+s-1)}(x), \tag{11}$$

$$x_+^{-r} \circ x_-^{-s} = \frac{(-1)^r c(\rho)}{(r+s-1)!}\delta^{(r+s-1)}(x), \tag{12}$$

$$F(x_+, -r) \circ F(x_-, -s) = (-1)^r \frac{c(\rho) + \frac{1}{2}\psi(s-1)}{(r+s-1)!}\delta^{(r+s-1)}(x), \tag{13}$$

for $r, s = 1, 2, \dots$.

Proof. We put

$$(x_-^{-s})_n = x_-^{-s} * \delta_n(x)$$

so that

$$(x_-^{-s})_n = -\frac{1}{(s-1)!}\int_x^{1/n} \ln(t-x)\delta_n^{(s)}(t)\,dt$$

on the interval $[0, 1/n]$, the intersection of the supports of $F(x_+, -r)$ and $(x_-^{-s})_n$. Then on using equation (1) we have

$$\langle F(x_+, -r), (x_-^{-s})_n x^k\rangle = \int_0^{1/n} x^{-r}\Big[\psi_k(x) - \sum_{i=0}^{r-1}\frac{x^i}{i!}\psi_k^{(i)}(0)\Big]\,dx, \tag{14}$$

where

$$\psi_k(x) = (x_-^{-s})_n x^k.$$

Now

$$\psi_k^{(i)}(x) = \begin{cases} cc\sum_{j=0}^{i}\binom{i}{j}\frac{k!}{(k-j)!}(x_-^{-s})_n^{(i-j)}x^{k-j}, & 0 \le i < k, \\ \sum_{j=0}^{k}\binom{i}{j}\frac{k!}{(k-j)!}(x_-^{-s})_n^{(i-j)}x^{k-j}, & i \ge k \end{cases}$$

Thus

$$\psi_k^{(i)}(0) = 0, \tag{15}$$

for $i = 0, 1, \dots, k-1$ and

$$\begin{aligned} \psi_k^{(i)}(0) &= \frac{i!}{(i-k)!}(x_-^{-s})_n^{(i-k)}\Big|_{x=0} \\ &= -\frac{i!}{(i-k)!(s-1)!}\int_0^{1/n}\ln t\,\delta_n^{(s+i-k)}(t)\,dt \qquad (16)\\ &= \frac{i!n^{s+i-k}}{(i-k)!(s-1)!}\int_0^1[\ln u - \ln n]\rho^{(s+i-k)}(u)\,du, \end{aligned}$$

for $i = k, k+1, \dots$, where the substitution $nt = u$ has been made. On making the substitution $nx = v$, it follows from equations (14), (15) and (16) that

$$\begin{aligned} \langle F(x_+, -r), (x_-^{-s})_n x^k\rangle &= n^{r-1}\int_0^1 v^{-r}\Big[\psi_k(v/n) - \sum_{i=k}^{r-1}\frac{v^i}{i!n^i}\psi_k^{(i)}(0)\Big]\,dv \\ &= -\frac{n^{r+s-k-1}}{(s-1)!}\int_0^1\Big[v^{k-r}\int_v^1[\ln(u-v) - \ln n]\rho^{(s)}(u)\,du + \\ &\quad - \sum_{i=k}^{r-1}\frac{v^{i-r}}{(i-k)!}\int_0^1[\ln u - \ln n]\rho^{(s+i-k)}(u)\,du\Big]\,dv \end{aligned}$$

and it follows that

$$\operatorname*{N-lim}_{n\to\infty}\langle F(x_+, -r), (x_-^{-s})_n x^k\rangle = 0, \tag{17}$$

for $k = 0, 1, \ldots, r+s-2$, the sum in the integral being empty if $k > r-1$. When $k = r+s-1$ we have

$$\langle F(x_+, -r), (x_-^{-s})_n x^{r+s-1} \rangle = -\frac{1}{(s-1)!} \int_0^1 v^{s-1} \int_v^1 [\ln(u-v) - \ln n] \rho^{(s)}(u)\, du\, dv$$

and on making the substitution $v = uy$, it follows that

$$\begin{aligned} \mathrm{N}\!-\!\lim_{n\to\infty} \langle F(x_+, -r), (x_-^{-s})_n x^{r+s-1} \rangle &= -\frac{1}{(s-1)!} \int_0^1 v^{s-1} \int_v^1 \ln(u-v) \rho^{(s)}(u)\, du\, dv \\ &= -\frac{1}{(s-1)!} \int_0^1 \rho^{(s)}(u) \int_0^u v^{s-1} \ln(u-v)\, dv\, du \\ &= -\frac{1}{(s-1)!} \int_0^1 u^s \ln u \rho^{(s)}(u) \int_0^1 y^{s-1}\, dy\, du + \\ &\quad -\frac{1}{(s-1)!} \int_0^1 u^s \rho^{(s)}(u) \int_0^1 y^{s-1} \ln(1-y)\, dy\, du. \end{aligned}$$

Now

$$\begin{aligned} \int_0^1 y^{s-1} \ln(1-y)\, dy &= s^{-1} \int_0^1 \ln(1-y)\, d(y^s - 1) \\ &= -s^{-1} \int_0^1 \frac{y^s - 1}{y - 1}\, dy = -s^{-1} \psi(s) \end{aligned}$$

and it is easily proved by induction that

$$\begin{aligned} \int_0^1 u^s \ln u \rho^{(s)}(u)\, du &= (-1)^s s! [c(\rho) + \tfrac{1}{2}\psi(s)], \\ \int_0^1 u^s \rho^{(s)}(u)\, du &= \tfrac{1}{2}(-1)^s s!. \end{aligned}$$

Thus

$$\mathrm{N}\!-\!\lim_{n\to\infty} \langle F(x_+, -r), (x_-^{-s})_n x^{r+s-1} \rangle = (-1)^{s+1} c(\rho). \tag{18}$$

Further, when $k = r+s$ we have

$$\begin{aligned} \langle F(x_+, -r), (x_-^{-s})_n x^{r+s} \rangle &= -\frac{n^{-1}}{(s-1)!} \int_0^1 v^{s-1} \int_v^1 [\ln(u-v) - \ln n] \rho^{(s)}(u)\, du\, dv \\ &= o(n^{-1} \ln n). \end{aligned}$$

Now let ϕ be an arbitrary function in $\mathcal{D}$. Then

$$\phi(x) = \sum_{k=0}^{r+s-1} \frac{x^k}{k!}\phi^{(k)}(0) + \frac{x^{r+s}}{(r+s)!}\phi^{(r+s)}(\xi x),$$

where $0 < \xi < 1$. It follows that

$$\begin{aligned}\langle F(x_+, -r)(x_-^{-s})_n, \phi(x)\rangle - \sum_{k=0}^{r+s-1} \frac{\phi^{(k)}(0)}{k!}\langle F(x_+, -r), (x_-^{-s})_n x^k\rangle \\ = \frac{1}{(r+s)!}\langle F(x_+, -r), (x_-^{-s})_n x^{r+s}\phi^{(r+s)}(\xi x)\rangle \\ = o(n^{-1}\ln n)\end{aligned} \tag{19}$$

since

$$\langle F(x_+, -r), (x_-^{-s})_n x^{r+s}\rangle = o(n^{-1}\ln n).$$

Thus

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty}\langle F(x_+, -r)(x_-^{-s})_n, \phi(x)\rangle &= \operatorname*{N-lim}_{n\to\infty} \sum_{i=0}^{r+s-1} \frac{\phi^{(k)}(0)}{k!}\langle F(x_+, -r), (x_-^{-s})_n x^k\rangle \\ &= \frac{(-1)^{s+1}c(\rho)\phi^{(r+s-1)}(0)}{(r+s-1)!} \\ &= \frac{(-1)^r c(\rho)}{(r+s-1)!}\langle \delta^{(r+s-1)}(x), \phi(x)\rangle\end{aligned}$$

on using equations (17), (18) and (19). Equation (12) follows.
Equation (10) follows from equations (3), (9) and (10), equation (11) follows from equations (4), (6) and (12) and equation (13) follows from equations (3), (4), (5), (6), (9) and (12).

Corollary. *The neutrix products* $F(x_-, -r) \circ x_+^{-s}$, $x_-^{-r} \circ F(x_+, -s)$, $x_-^{-r} \circ x_+^{-s}$ *and* $F(x_-, -r) \circ F(x_+, -s)$ *exist and*

$$F(x_-, -r) \circ x_+^{-s} = \frac{(-1)^{s+1}r!}{2(r+s)!}\delta^{(r+s-1)}(x), \tag{20}$$

$$x_-^{-r} \circ F(x_+, -s) = (-1)^{s+1}\frac{c(\rho) + \frac{1}{2}\psi(s-1)}{(r+s-1)!}\delta^{(r+s-1)}(x), \tag{21}$$

$$x_-^{-r} \circ x_+^{-s} = \frac{(-1)^{s-1}c(\rho)}{(r+s-1)!}\delta^{(r+s-1)}(x), \tag{22}$$

$$F(x_-, -r) \circ F(x_+, -s) = (-1)^{s+1}\frac{c(\rho) + \frac{1}{2}\psi(s-1)}{(r+s-1)!}\delta^{(r+s-1)}(x), \tag{23}$$

for $r, s = 1, 2, \ldots$.

Proof. Equation (22) follows from equation (12) on replacing x by $-x$. Equation (20) then follows from equations (4), (7) and (22), equation (21) follows from equations (3), (8) and (22) and equation (23) follows from equations (3), (4), (5), (7), (8) and (22).

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Received 01.03.1993