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Strong Solvability of a Boundary Value Problem for an Equation of Mixed Type

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In this paper a boundary value problem for an equation of mixed type is considered. It is proved that the weak solution is a strong one too and the strong solution is unique.

Let G be a bounded domain in the plane (x, y) with a partially smooth boundary $\Gamma \cup \sigma_1 \cup \sigma_2$ (fig.1), where Γ_0 is a two-times smooth curve and σ_1 , σ_2 are characteristic of equation:

(1) $$Lu \equiv u_{xx} + k(y)u_{yy} + \alpha_1(x, y)u_x + \alpha_2(x, y)u_y + \gamma(x, y)u = f(x, y) ,$$

where $k(y) \in C^2(\overline{G})$, $y\, k(y) > 0$ for $y \neq 0$, $\alpha_1(x, y)$ and $\alpha_2(x, y) \in C^1(\overline{G})$ and $\gamma(x, y) \in C(\overline{G})$.

For $y > 0$ equation (1) is elliptic, for $y = 0$ it is parabolic and for $y < 0$ - hyperbolic.

We consider the following problem:

Problem A. Find a solution of equation (1) satisfying the boundary value conditions:

(2) $$u(x, y) = 0 \text{ on } \Gamma_0$$

$$u(x, y)_{|\sigma_1 \cup \sigma_2} \quad \sim$$

(the sign $\sim$ denotes that no boundary value conditions are prescribed).

The conjugate boundary value condition for equation (1) are:

(3) $$v = 0 \text{ on } \Gamma \cup \sigma_1 \cup \sigma_2$$

Denote by $W_2^2(\overline{G})$ and $W_{2,*}^2(\overline{G})$ the closure with respect to the norm $W_2^2(G)$ of the set of functions of $C^2(\overline{G})$, satisfying (2) and (3).

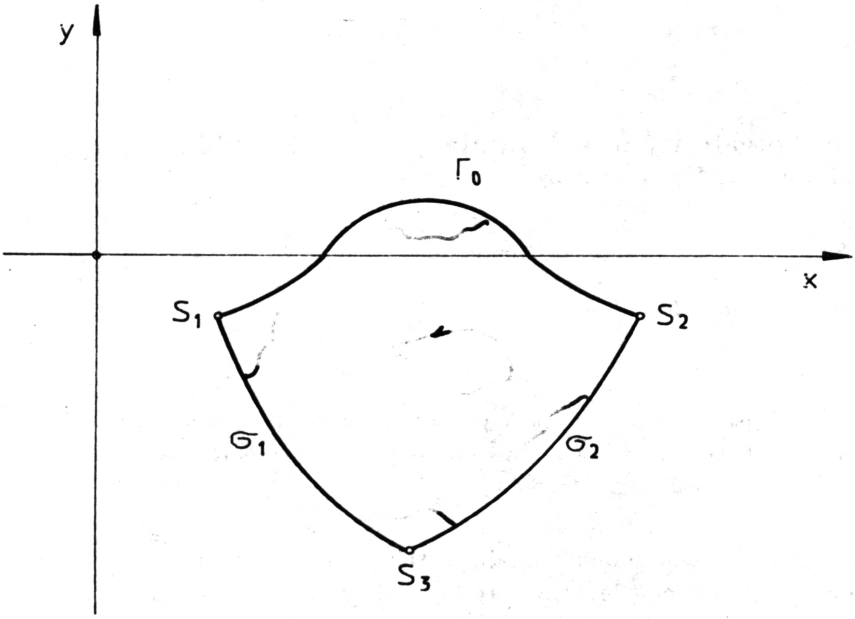

Fig. 1.

Definition. The function $u \in L_2(G)$ is said to be a weak solution of problem A if $(u, L^*v) = (f, v)$ for each $v \in W_{2,*}^2(\overline{G})$. ($L^*$ is a formally conjugate of L operator).

Definition. The function $u \in L_2(G)$ is said to be a strong solution of problem A, if there exists a sequence $u^k \in W_2^2(\overline{G})$,

$$\|u^k - u\| \to 0, \ \|Lu^k - f\| \to 0, \ k \to \infty \ .$$

Let $b(x, y) = b_1 + b_2 y$ be a function, whose coefficients satisfy the following conditions:

a). b_2 is a fixed arbitrary positive number;

b). b_1 satisfies conditions $b_1 + b_2 y > 0$ for $y \in [h_1, h_2]$ ($h_1 = \min_{\overline{G}} y$,

$h_2 = \max_{\overline{G}} y$), and

$$(2\alpha_2 - k')b_1 - kb_2 \geq 2M = \text{const} \; > 0 \quad \in \overline{G} \, ,$$

simultaneously, with the additional requirement

$$2\alpha_2 - k' \geq \tau = \text{const} \; > 0 \quad \in \overline{G} \, .$$

Let $a = -1/(\tau y + \delta)$, where $\delta = -1 - \tau h_2$.

The function $\alpha_1(x, y)$ and $\gamma(x, y)$ are chosen sufficiently small in absolute value (see [1]).

Let $n_y > 0$, $kn_y^2 + n_x^2 > 0$ and $an_x + bn_y > 0$ on Γ_0 ((n_x, n_y) is the unit vector of outer normal towards Γ_0).

Under the above assumption, we have proved that problem A has a weak solution $u(x, y) \in W_2^1(G) \subset L_2(G)$ for each function $f \in L_2(G)$, as $u(x, y) = 0$ almost everywhere on Γ_0, i.e.:

$$||u||_{L_2(\Gamma_0)} = 0$$

Now we prove the following theorem.

Theorem 1. *The weak solution of problem A is also a strong solution.*

Proof. Let $\{\phi_i(x, y)\}$, $i = 1, 2, \ldots, m$ be a separation of the unity for the domain G i.e. $\phi_i(x, y) \in C_0^\infty(\Omega_i)$, where Ω_i are open sets so that $G \in \cup \Omega_i$ and

$$\sum_{i=1}^{m} \phi_i(x, y) = 1 \text{ in } G \, .$$

If the function $u(x, y) \in W_2^1(G)$ is a weak solution of problem (1) and (2), then the function $w_i = \phi_i u$ is a weak solution of the following problem

$$Lw_i = g_i \;, \; g_i \in L_2(G) \;, \; i = 1, 2, \ldots, m \;, \tag{4}$$

where

$$g_i = \phi_i f + uL\phi_i + 2\frac{\partial u}{\partial x} \cdot \frac{\partial \phi_i}{\partial x} + 2k\frac{\partial u}{\partial y} \cdot \frac{\partial \phi_i}{\partial y} - \gamma u \phi_i$$

and the boundary value condition:

$$(5) \qquad w_i = 0 \text{ on } \Gamma_0$$

If one proves that the functions w_i, $i = 1, 2, \ldots, m$ are strong solutions of problem (4) and (5), then the function

$$u = \sum_{i=1}^{m} w_i$$

is to be a strong solution of problem A. Following the manner of [2, 5] we shall establish, that the weak solution $u(x, y)$ is the strong one, by proving that w_i are strong solutions.

Let $j(t)$ is a function infinitelly time differentiable in the interval $-\infty < t < \infty$, so that $j(-t) = j(t)$; $j(t) > 0$ for $|t| < 1$; $j(t) = 0$ for $|t| \geq 1$ and $\int_{-1}^{1} j(t)dt = 1$.

For the geometrical disposition of supp w_i the following five cases are possible:

1). supp $w_i \cap \Gamma = \emptyset$.

In this case we suppose that Ω_i is an inner neighbourhood. Let ε be an arbitrary positive number. We put

$$j_\varepsilon(x, y) = \frac{1}{\varepsilon^2} j(\frac{x}{\varepsilon}) j(\frac{y}{\varepsilon})$$

and consider the average operator:

$$J_\varepsilon w_i = \int_{\Omega_i} j_\varepsilon(x - \bar{x}, y - \bar{y}) w_i(\bar{x}, \bar{y}) d\bar{x} d\bar{y};$$

It is known [2, 3] that $J_\varepsilon w_i \in C_0^\infty(\Omega_i)$ and $\|J_\varepsilon w_i - w_i\|_{L_2(\Omega_i)} \to 0$ for $\varepsilon \to 0$.

For an arbitrary function $v(x, y) \in C_0^\infty(\Omega_i)$ and sufficiently small $J_\varepsilon^* v \in C_0^\infty(\Omega_i)$. Then, the definition of a weak solution gives:

$$(w_i, (L^* J_\varepsilon^*) v)_{L_2(\Omega_i)} = (g_i, J_\varepsilon^* v)_{L_2(\Omega_i)}$$

whence

$$((L^* J_\varepsilon^*)^* w_i, v)_{L_2(\Omega_i)} = (J_\varepsilon g_i, v)_{L_2(\Omega_i)}$$

and it follows that $(L^* J_\varepsilon^*)^* w_i = J_\varepsilon g_i$. It is proved that:

$$\|L J_\varepsilon w_i - (L^* J_\varepsilon^*)^* w_i\|_{L_2(\Omega_i)} \to 0 \ ,$$

from where

$$\|LJ_\varepsilon w_i - g_i\|_{L_2(\Omega_i)} \le \|LJ_\varepsilon w_i - (L^*J_\varepsilon^*)^* w_i\|_{L_2(\Omega_i)} + \|(L^*J_\varepsilon^*)^* w_i - g_i\|_{L_2(\Omega_i)}$$
$$= \|LJ_\varepsilon w_i - (L^*J^*)^* w_i\|_{L_2(\Omega_i)} + \|J_\varepsilon g_i - g_i\|_{L_2(\Omega_i)} \to 0 \text{ as } \varepsilon \to 0 .$$

Therefore $w_{i\varepsilon} = J_\varepsilon w_i$ is the approximation sequence of the strong solution.

2). supp $w_i \cap \Gamma_0 = \Gamma_i'$: supp $w_i \cap (\sigma_1 \cup \sigma_2) = \emptyset$

Since Γ_0 is a two-times smooth and noncharacteristic curve, then there exists a C^2- smooth and nonsingular transformation of the independent variables:

$$x_1 = x_1(x,y) \ , \ y_1 = y_1(x,y) \ ,$$

which transforms $\Omega_i' = \Omega_i \cap G$ into a Ω_i'' neighbourhood of the semispace $y_1 \ge 0$ and Γ_i' goes into Γ_i'', a part of the line $y_1 = 0$.

Under this transformation, the coefficient associated with $\dfrac{\partial^2 \tilde{w}_i}{\partial y_1^2}$, is different from zero on the line $y_1 = 0$ and in a neighbourhood of it. Equation (4) can be reduced to the form:

$$\tilde{L}\tilde{w}_i = \frac{\partial^2 \tilde{w}_i}{\partial y_1^2} + M_1 \frac{\partial \tilde{w}_i}{\partial y_1} + M_2 \tilde{w}_i = \tilde{g}_i \ , \tag{6}$$

where M_1 and M_2 are differential operators, respectively of first and second order, involving derivatives only with respect to x_1. Since $\tilde{w}_i$ is not defined outside of Ω_i'', then problem (6) and (5) can be extended to the whole semispace $y_1 \ge 0$ (denoted by R_2^+), in the following way:

$$\tilde{w}_i = 0 \ , \ \tilde{g}_i = 0 \ , \ \text{outside of } \Omega_i'' \ ,$$

the coefficient associated with $\dfrac{\partial^2 \tilde{w}_i}{\partial y_1^2}$, is put to be 1 for $y_1 \ge 0$ and the coefficients of M_1 and M_2 are extended by keeping their smoothness. The boundary value condition has the previous form:

$$\tilde{w}_i = 0 \text{ for } y_1 = 0 \tag{7}$$

Let $j_\varepsilon^t = j(x/\varepsilon)/\varepsilon$ and let us consider the tangent average operator

$$J_\varepsilon^t \tilde{w}_i = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} j_\varepsilon^t(x_1 - \bar{x}_1) \tilde{w}_i(\bar{x}_1, y_1) d\bar{x}_1 \ .$$

It is known [3,5] that the operator J_ε^t possesses the following properties:

a) For each function $u \in L_2(R_2^+)$

$$\|J_\varepsilon^t u - u\|_{L_2(R_2^+)} \to 0 \text{ for } \varepsilon \to 0 .$$

b) if

$$M_1 = a\frac{\partial}{\partial x_1} + b ,$$

where a is a partially differentiable function and b is partially continuous, then the operator $M_1 J_\varepsilon^t - J_\varepsilon^t M_1$ is uniformly bounded in $L_2(R_2^+)$ and for every function $u \in L_2(R_2^+)$

$$\|(M_1 J_\varepsilon^t - J_\varepsilon^t M_1)u\|_{L_2(R_2^+)} \to 0 \text{ as } \varepsilon \to 0 .$$

c) If

$$M_2 = a\frac{\partial^2}{\partial x_1^2} + b\frac{\partial}{\partial x_1} + c ,$$

where, a is a two times smooth, b is a smooth and c is a partially continuous function, then the operator $M_2 J_\varepsilon^t - J_\varepsilon^t M_2$ is uniformly bounded in $W_2^1(R_2^+)$ and for every function $u \in W_2^1(R_2^+)$

$$\|(M_2 J_\varepsilon^t - J_\varepsilon^t M_2)u\|_{L_2(R_2^+)} \to 0 \text{ as } \varepsilon \to 0 .$$

The property a) implies that

$$\|J_\varepsilon^t \tilde{w}_i - \tilde{w}_i\|_{L_2(R_2^+)} \to 0 \text{ for } \varepsilon \to 0 . \tag{8}$$

Let $v(x_1, y_1)$ be arbitrary function of $C_0^\infty(R_2^+)$, then for a sufficiently small $\varepsilon > 0$, $J_\varepsilon^{t*} v \in C_0^\infty(R_2^+)$ and the definition of a weak solution gives:

$$(\tilde{w}_i, (\tilde{L}^* J_\varepsilon^{t*})v)_{L_2(R_2^+)} = (\tilde{g}_i, J_\varepsilon^{t*} v)_{L_2(R_2^+)} = (J_\varepsilon^t \tilde{g}_i, v)_{L_2(R_2^+)}$$

Here $J_\varepsilon^{t*} = J_\varepsilon^t$ and having in mind that $\frac{\partial}{\partial y_1}$ and $\frac{\partial^2}{\partial y_1^2}$ commute with J_ε^t, we have

$$\tilde{L} J_\varepsilon^t \tilde{w}_i = J_\varepsilon^t \tilde{g}_i + (M_1 J_\varepsilon^t - J_\varepsilon^t M_1)\frac{\partial \tilde{w}_i}{\partial y_1} + (M_2 J_\varepsilon^t - J_\varepsilon^t M_2)\tilde{w}_i .$$

This equality is to be understood in a sense of a functional defined on the functions of $C_0^\infty(R_2^+)$. From the properties listed above for J_ε^t, it follows that $LJ_\varepsilon^t w_i \in L_2(R_2^+)$ and

$$(9)\quad \begin{aligned} \|\tilde{L}J_\varepsilon^t\tilde{w}_i - \tilde{g}_i\|_{L_2(R_2^+)} &\le \|J_\varepsilon^t\tilde{g}_i - \tilde{g}_i\|_{L_2(R_2^+)} + \|(M_1J_\varepsilon^t - J_\varepsilon^t M_1)\frac{\partial\tilde{w}_i}{\partial y_1}\|_{L_2(R_2^+)} \\ &+\|(M_2J_\varepsilon^t - J_\varepsilon^t M_2)\tilde{w}_i\|_{L_2(R_2^+)} \qquad \text{for } \varepsilon \to 0 \end{aligned}$$

The functions $\tilde{w}_{i\varepsilon} = J_\varepsilon^t\tilde{w}_i$ have derivatives of all orders with respect to x_1 belonging to $L_2(R_2^+)$. Since $\tilde{w}_i \in W_2^1(R_2^+)$ then $\frac{\partial\tilde{w}_{i\varepsilon}}{\partial y_1} \in L_2(R_2^+)$. The derivatives $\frac{\partial^2\tilde{w}_{i\varepsilon}}{\partial x_1\partial y_1} \in L_2(R_2^+)$ and $\frac{\partial^2\tilde{w}_i}{\partial y_1^2} = \tilde{L}\tilde{w}_i - M_1\frac{\partial\tilde{w}_{i\varepsilon}}{\partial y_i} - M_2\tilde{w}_{i\varepsilon} \in L_2(R_2^+)$ obviously exist, i.e. $\tilde{w}_{i\varepsilon} \in W_2^2(R_2^+)$. It is verified that $\tilde{w}_{i\varepsilon} = 0$ for $y_1 = 0$ and (8) and (9) imply that $\tilde{w}_i$ is a strong solution of problem (6), (7).

3). supp $w_i \cap \sigma_1 = \Gamma_i'$ (or $w_i \cap \sigma_2 = \Gamma_i'$).

First, as in case 2), we transform the problem (4) and (5). Since Γ_i' is on the characteristic, then equation (4) takes the form:

$$(10)\quad \tilde{L}\tilde{w}_i = a_{11}\frac{\partial^2\tilde{w}_i}{\partial y_1^2} + a_{12}\frac{\partial^2\tilde{w}_i}{\partial y_1\partial x_1} + a_{22}\frac{\partial^2\tilde{w}_i}{\partial x_1^2} + b_1\frac{\partial\tilde{w}_i}{\partial y_1} + b_2\frac{\partial\tilde{w}_i}{\partial x_1} + c\tilde{w}_i = \tilde{g}_i$$

with $a_{11}(x_1, y_1) = 0$ for $y_1 = 0$ and

$$(11)\qquad \tilde{w}_i| \quad \sim \qquad \text{for } y_1 = 0$$

We put

$$J_\varepsilon^1 w_i = \frac{1}{\varepsilon}\int_{R_2^+} j(\frac{y_1 - \bar{y}_1}{\varepsilon} + 2)j_\varepsilon^t(x_1 - \bar{x}_1)\tilde{w}_i(\bar{x}_1, \bar{y}_1)d\bar{x}_1 d\bar{y}_1\ ;$$

As it is known from [2,3] $J_\varepsilon^1\tilde{w}_i \in C_0^\infty(R_2^+)$ and $\|J_\varepsilon^1\tilde{w}_i - \tilde{w}_i\|_{L_2(R_2^+)} \to 0$ for $\varepsilon \to 0$.

If v is an arbitrary function of $L_2(R_2^+)$, then $J_\varepsilon^1 w_i \in C^\infty(R_2^+)$ and it satisfies the conjugate boundary value conditions

$$J_\varepsilon^{1*}v = 0 \qquad \text{for } y_1 = 0\ .$$

From the definition of a weak solution, we have

$$(\tilde{w}_i, (L^* J_\varepsilon^{1*})v)_{L_2(R_2^+)} = (\tilde{g}_i, J_\varepsilon^{1*} v)_{L_2(R_2^+)}$$

and we can follow further the pattern of case 1). And so

$$\|\tilde{L} J_\varepsilon^1 \tilde{w}_i - \tilde{g}_i\|_{L_2(R_2^+)} \to 0 \qquad \varepsilon \to 0 \, .$$

Therefore $\tilde{w}_i$ is a strong solution of the problem (10) and (11).

4). supp $w_i \cap \Gamma_0 = \Gamma_i'$, supp $w_i \cap \sigma_1 = \Gamma_i''$ (or supp $w_i \cap \sigma_2 = \Gamma_i''$).

Denote $\Omega_i' = \Omega_i \cap G$. There exists a C^2-smooth and nonsingular transformation $x_1 = x_1(x,y)$, $x_1 = y_1(x,y)$ mapping the neighbourhood Ω_i' into a neighbourhood Ω_i'' of the angle $[x_1 \geq 0 \, , \, y_1 \geq 0]$ such that Γ_i' goes into Γ_{i0}' (Γ_{i0}' is on $y_1 = 0$) and Γ_i'' goes into Γ_{i1}'', where Γ_{i1}'' is a part of the line $x_1 = 0$. Since Γ_0 is not a characteristic curve, then the equation (4) takes the form (6). We extend the problem in the space: $R_2' = \{y_1 \geq 0, \; x_1 \geq 0\}$. The boundary value condition is $\tilde{w}_i = 0$ for $y_1 = 0$ and the conjugate one is $v = 0$ for $x_1 = 0$. We continue then as in the case 2). The approximation sequence is:

$$J_\varepsilon^{t_1} \tilde{w}_i = \frac{1}{\varepsilon} \int_0^\infty j(\frac{x_1 - \bar{x}_1}{\varepsilon} + 2) \tilde{w}_i(\bar{x}_1, y_1) d\bar{x}_1 \; ;$$

5). supp $w_i \cap \sigma_1 = \Gamma_i'$ and supp $w_i \cap \sigma_2 = \Gamma_i''$.

First, as in the previous case, we transform the problem. Since σ_1 is a characteristic, then the equation of problem (4) takes form (10).

The boundary value condition is

$$\tilde{w}_{i_{|y_1=0}} = \quad \sim \qquad \tilde{w}_{i_{|x_1=0}} = \quad \sim \; . \tag{12}$$

We put

$$J_\varepsilon^2 \tilde{w}_i = \frac{1}{\varepsilon^2} \int_{R_2'} j(\frac{x_1 - \bar{x}_1}{\varepsilon} + 2) j(\frac{y_1 - \bar{y}_1}{\varepsilon} + 2) \tilde{w}_i(\bar{x}_1, \bar{y}_1) d\bar{x}_1 d\bar{y}_1 \; ;$$

It follows

$$\|J_\varepsilon^2 \tilde{w}_i - \tilde{w}_i\|_{L_2(R_2')} \to 0 \qquad \text{for } \varepsilon \to 0 \, .$$

If v is an arbitrary function of $L_2(R_2')$, then $J_\varepsilon^{2*} v \in C_0^\infty(R_2')$ and satisfies the conjugate boundary value conditions

$$J_\varepsilon^{2*} v = 0 \qquad \text{for } y_1 = 0 \text{ and } x_1 = 0 \, .$$

We continue then as in case 1). The approximation sequence of a strong solution of a problem (10) and (12) is $\tilde{w}_{i\varepsilon} = J^2_\varepsilon \tilde{w}_i$. This completes the proof of the theorem.

In [1] we have shown that for the function $u(x,y) \in W^1_2(G)$ satisfying equation (1) and boundary value condition (2) the a priori estimation

$$\|Lu\|_{L_2(G)} \geq C\|u\|_{W^1_2(G)} \tag{13}$$

holds.

From (13) and theorem 1 next theorem follows:

Theorem 2. *The boundary value problem A has a unique strong solution.*

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