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Characterization of the Smoothest Periodic Interpolant

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This paper is devoted to the problem of characterizing the periodic function of minimum L_p -norm of its rth derivative, taking consequently given values $(y_i)_1^{2N}$.

1. Introduction

Let $y_1,..,y_{2N}$ be given real fixed data, satisfying the requirement

$$(1)\qquad (y_i - y_{i-1})(y_{i+1} - y_i) < 0,\ i = 1,..,2N,\ (y_{2N+1} = y_1)$$

and

$$\Sigma_{2N} = \{t\, t = (t_1,..,t_{2N}),\ 0 \le t_1 < .. < t_{2N} < 1\}.$$

For $p \in (1,\infty]$, $\tilde{W}_p[0,1]$ $(r \ge 2)$ denotes the usual Sobolev space of periodic real-valued functions on $[0,1]$ with $r-1$ absolutely continuous derivatives and r-th derivative existing almost everywhere as a function in $L_p[0,1]$. We assume here that the period is equal to 1 and that $2N > r$. With any $t \in \Sigma$ and $\{y_k\}_1^{2N}$, we associate the set of functions

$$\tilde{W}_p^r(t;y) = \left\{f\ f \in \tilde{W}_p^r[0,1],\ f(t_i) = y_i,\ i = 1,..,2N\right\}.$$

In this work, we are interested in the solution of the extremal problem

$$(2)\qquad \inf_{t\in\Sigma_{2N}} \quad \inf\left\{\|f^{(r)}\|_p\ f \in \tilde{W}_p^r(t;y)\right\}.$$

In Section 2 and 3 we give a complete characterization of the solution to (2) for $p \in (1,\infty]$. The pair (t^*, f^*), $t^* \in \Sigma_{2N}$, $f^* \in \tilde{W}_p^r[0,1]$, for which both infima in (2) are attained is called a solution of the extremal problem (2). We prove that in the case $p = \infty$ the solution (t^*, f^*) is unique. In Section 4 we consider the case $r = 2$, $p \in (1, \infty]$. First we prove uniqueness of the solution (t^*_p, f^*_p) and find explicitly the extremal function when $p \in (1, \infty)$. Letting p tend to ∞ we show that $t^*_\infty = \lim_{p\to\infty} t^*_p$, $f^*_\infty = \lim_{p\to\infty} f^*_p$ form the unique solution of (2) in $\tilde{W}^2_\infty[0,1]$. An interesting feature of the case $r = 2$, $p \in (1, \infty]$ is that the extremal nodes t do not depend on parameter p.

The problem (2) in the nonperiodic case is discussed by A. Pinkus [2] and in more general case by B. Bojanov [8]. The results connected with the uniqueness of the extremal function in the nonperiodic case are known only for $p = 2$, $r = 2$ (S. Marin [6]), $p \in (1, \infty)$, $r = 2$ and $p = 2$, $r = 3$ (R. Uluchev [5]), $p = \infty$ (A. Pinkus [2], B. Bojanov [8],).

2. Characterization of the solution to (2) for $p \in (1, \infty)$

Let $t \in \Sigma_{2N}$ be fixed. $f[t_i, .., t_{i+r}]$ denotes the rth divided difference of f at the points $t_i, .., t_{i+r}$, $i = 1, .., 2N$ $(t_{2N+j} = t_j + 1,\ j = 1, .., r)$. For $f \in \tilde{W}^r_p(t; y)$, set

$$D_i = f[t_i, .., t_{i+r}],\ i = 1, .., 2N,\ (f(t_{2N+j}) = y_j,\ j = 1, .., r).$$

We have

$$D_i = \int_0^1 \tilde{B}_{i,r}(x) f^{(r)}(x)dx,\ i = 1, .., 2N,$$

where $\tilde{B}_{i,r}(x)$ is the periodic B-spline of degree $r - 1$ with knots $t_i, .., t_{i+r}$, $i = 1, .., 2N$. Let us consider the problem

$$\inf\left\{\|f^{(r)}\|_p\ f \in \tilde{W}^r_p(t; y)\right\}. \tag{3}$$

We set $g(x) = f^{(r)}(x)$. Problem (3) is equivalent to

$$\inf\left\{\|g\|_p\ \int_0^1 \tilde{B}_{i,r}(x)g(x)dx = D_i,\ i = 1, .., 2N\right\}. \tag{4}$$

It is known (see B. Bojanov [7]) that problem (4) has a unique solution. It has the form

$$g(x) = |\sum_{i=1}^{2N} \alpha_i \tilde{B}_{i,r}(x)|^{q-1} \mathrm{sign}(\sum_{i=1}^{2N} \alpha_i \tilde{B}_{i,r}(x)), \tag{5}$$

where $1/p + 1/q = 1$, and

$$D_i = \int_0^1 \tilde{B}_{i,r}(x)g(x)dx, \; i = 1,..,2N. \tag{6}$$

Equation (6) determines uniquely the coefficients $(\alpha_i)_1^{2N}$ in expression (5).

The existence and characterization of the solution to (2) in the case $p \in (1, \infty)$ is given by the following

Theorem 2.1. *Given N, r and $\{y_i\}_1^{2N}$, satisfying (1) there exists a solution (t^*, f^*) to the problem (2). Furthermore,*

$$f^{*(r)} = |\sum_{i=1}^{2N} \alpha \tilde{B}_{i,r}(x)|^{q-1} \mathrm{sign}(\sum_{I=1}^{2N} \alpha \tilde{\mathrm{B}}_{\mathrm{i,r}}(\mathrm{x})),$$

where $1/p + 1/q = 1$, $\tilde{B}_{i,r}(x)$ is the periodic B-spline of degree $r-1$ with knots $t_i^*, .., t_{i+r}^*$, and the $\{\alpha_i\}_1^{2N}$ satisfy the equations

$$\int_0^1 B_{i,r}(x) f^{*(r)}(x)dx = f[t_i^*, .., t_{i+r}^*], \; i = 1,..,2N, \tag{7}$$

and

$$f^{*'}(t_i^*) = 0, \; i = 1,..,2N. \tag{8}$$

P r o o f. Let f^* solve (2). Existence implies that there is a $t \in \Sigma_{2N}$ for which $f^* \in \tilde{W}_p^r(t^*; y)$. Since f^* must also solve (3) for t^*, it follows that f necessarily satisfies (7). It remains to prove that (8) holds.

It follows by the character of the data that, $f^{*'}$ has at least $2N$ distinct zeros. From Rolle's theorem applied to $f^{*'}$, we obtain that $f^{*(r)}$ has at least $2N$ distinct zeros. On the other hand, by Theorem 8.4 of L. S h u m a k e r [9] (about the estimation of the number of zeros of periodic splines) $f^{*(r)}$ has at most $2N$ zeros. Hence, $f^{*(r)}$ has exactly 2N distinct zeros. This entails that $f^{*'}$ has exactly $2N$ distinct zeros too.

Let $0 \le s_1 < ... < s_{2N} < 1$ denote the all extrema of f^*. It remains to prove that $s_i = t_i^*$, $i = 1,..,2N$, $(s_{2N+1} = s_1)$. Note that $t_{j-1}^* < s_j < t_{j+1}^*$, $j = 2,..,2N+1$. Assume that $s_i \neq t_i^*$ for some $i \in \{1,..,2N\}$. Consider the problem (3) at points $\{s_i\}_1^{2N}$ with values $y_i' = f^*(s_i)$, $i = 1,..,2N$. There is a unique solution to this new problem which we denote by g^*. Since, $g^* \not\equiv f^*$ we have

$$\|g^*\|_p < \|f^*\|_p. \tag{9}$$

Moreover, using that $\{s_i\}_1^{2N}$ are the extrema of f^* we obtain that there exist points $0 \le \eta_1 < .. < \eta_{2N} < 1$ such that

$$g^*(\eta_i) = y_i, \ i = 1, .., 2N. \tag{10}$$

Then (9) and (10) lead to contradiction with the minimality property of f^*. The proof is completed. ■

3. Characterization of the solution to (2) for $p = \infty$

Denote by $D_r(x)$ the periodic extension of the Bernoulli polynomial of degree r i.e.,

$$D_r(x) = 1/(2^{r-1}\pi^r)\sum_{\nu=1}^{\infty}(1/\nu^r)\cos(2\pi\nu x - \pi r/2).$$

A periodic perfect spline $P(x)$ of degree r with knots $\{\xi_i\}_1^{2N}$ is any expression of the form

$$c + 2\sum_{i=1}^{2N}(-1)^i D_{r+1}(x - \xi_i), \tag{11}$$

where c is a real number and the knots ξ_i are supposed to satisfy the restrictions

$$0 \le \xi_1 < .... < \xi_{2N} < \xi_{2N+1} = 1 + \xi_1, \quad \sum_{i=1}^{2N}(-1)^i(\xi_{i+1} - \xi_i) = 0.$$

We suppose that $2N > r$ and $t_1 = 0$. Denote by $\tilde{P}_{N,r}$ the set of all periodic perfect splines of degree r with $2N$ knots and by $P(N, r)$ the class of those $P \in \tilde{P}_{N,r}$ which have $2N$ distinct zeros in $[0, 1)$.

We will use the following auxiliary lemmas.

Lemma 3.1. *Given the numbers $\{t_i\}_1^{2N}$ satisfying*

$$0 \le t_1 < .. < t_{2N} < 1, \tag{12}$$

there exists a unique $P \in \tilde{P}_{N,r}$ such that

$$P(t_i) = 0, \ i = 1, .., 2N \tag{13}$$

$$(-1)P(x) > 0 \ in \ (t_1, t_2). \tag{14}$$

This fact is proved in [1].

Lemma 3.2. *Let* $0 \le \xi_1 < .. < \xi_k < \xi_{k+1} = 1 + \xi_1$ *be given points and*

$$u_1(x) = 1,\ u_j(x) = D_r(x - \xi_{j-1}) - D_r(x - \xi_k),\ j = 2, .., k,$$

Then

(1) The sequence $(u_j(x))_1^k$ *is a basis in the space of periodic spline functions of degree* $r - 1$ *with* $\xi_1, .., \xi_k$.

(2) For fixed $k = 2\nu + 1$ *there exists a number* $b = 1$ *or* $b = -1$ *such that the inequality*

$$\beta.D\begin{pmatrix} t_1, & .., & t_k \\ u_1, & .., & u_k \end{pmatrix} = \beta \det(u_i(t_j)) \ge 0$$

holds for every choice of $(t_i)_1^{2\nu+1}$ *and* $(\xi_i)_1^{2\nu+1}$ *satisfying*

$$0 \le t_1 < ... < t_{2\nu+1} < 1 + t_1,\ t_i < t_{i+r},\ i = 1, .., 2\nu + 1,$$

$$0 \le \xi_1 < ... < \xi_{2\nu+1} < \xi_{2\nu+2} = 1 + \xi_1$$

Furthmore, the strict inequality holds if and only if

$$t_i < \xi_i^* < t_{i+r},\ i = 1, .., 2\nu + 1,$$

where $\xi_1^*, ..., \xi_{2\nu+1}^*$ *is some cyclic arrangement of* $\xi_1, , .., \xi_{2\nu+1}$.

The proof can be found in [3].

Theorem 3.1. *Given* N, r *and the positive numbers* $e_1, \ldots, e_{2n}$, *there exists a unique perfect spline* $P \in P(N, r-1)$ *and a number* $R > 0$ *such that*

$$R \int_{t_i^*}^{t_{i+1}^*} P(x)dx = (-1)^i e_i,\ i = 1, .., 2N, \tag{15}$$

where $\{t_i^*\}_1^{2N}$ *are the zeros of* $P(x)$ *and* $0 = t_1^* < ... < t_{2N}^* < t_{2N+1} = 1$.

Moreover R *is a continuous strictly increasing function of* $e_1, .., e_{2N}$, *in the domain* $e_1 > 0, .., e_{2N} > 0$.

Proof. Let P_0 be an arbitrary element of $P(N, r-1)$. Suppose that

$$P_0(x) = c_0 + 2\sum_{i=1}^{2N}(-1)^i D_r(x - \xi_i^c),$$

where

$$(16) \qquad 0 \le \xi_1^0 < .... < \xi_{2N}^0 < \xi_{2N+1}^0 = 1 + \xi_1^0, \ \sum_{i=1}^{2N}(-1)^i(\xi_{i+1}^0 - \xi_i^c) = 0.$$

Let $(t_i^c)_1^{2N}$ be the zeros of P_0, $0 = t_1^0 < ... < t_{2N}^0 < 1$.
Denote further

$$e_i^c = |\int_{t_i^0}^{t_{i+1}^0} P_0(x)dx|, \ i = 1, .., 2N,$$

$$e_i(s) = e_i^c + s(e_i - e_i^c), \ i = 1, .., 2N, \ s \in [0, 1], \ e_i(s) > 0.$$

We construct a function $P(s, x) \in P(N, r-1)$ with parameters $t_i(s)$, $\xi_i(s)$, $c(s)$, and another function $R(s)$ such that

$$(17) \qquad R(s)\int_{t_i(s)}^{t_{i+1}(s)} P(s, x)dx = (-1)^i e_i(s), \ i = 1, .., 2N.$$

The parameters of $P(s, .)$ satisfy the system of equations

$$(18) \qquad \begin{cases} P(s, t_1(s)) = 0 \\ R(s) = \int_{t_i(s)}^{t_{i+1}(s)} P(s, x)dx = (-1)^i e_i(s), \ i = 1, .., 2N \\ \sum_{i=1}^{2N}(-1)^i(\xi_{i+1}(s) - \xi_i(s)) = 0 \\ P(s, t_i(s)) = 0, \ i = 2, ..., 2N \end{cases}$$

Evidently $(R_0 = 1, P_0)$ is a solution of (18).
Denote by $\Delta(s)$ the Jacobian of (18) with respect to R, c, $(\xi_i)_1^{2N}$, $(t_i)_2^{2N}$. $\Delta(s)$ has the form

	R	c	$\xi_1.....\xi_{2N}$	$t_2............t_{2N}$
1 . . $2N+1$				0
$2N+2$	0	0	$2 - ...... - 2$	0
$2N+3$ . . $4N+1$				D_1

where

$$D_1 = diag\left\{P'(s, t_2(s), ..., P'(s, t_{2N}(s)\right\}.$$

Now we annihilate the $(2N+2)$-st element of the columns, corresponding to $\xi_1, ..., \xi_{2N-1}$, adding that one, corresponding to ξ_{2N}, with an appropriate sign. Then

$$\det \Delta(s) = \prod_{i=2}^{2N} P'(s, t_i(s)). \det J(s), \tag{19}$$

where $J(s)$ is the matrix with rows

$$0, 1, (-1)^j[D_{r-1}(t_1(s) - \xi_{2N}(s)) - D_{r-1}(t_1(s) - \xi_j(s)],\ j = 1, .., 2N-1$$

$$\int_{t_i(s)}^{t_{i+1}(s)} P(s, \tau_i) d\tau_i,\ R(s) \int_{t_i(s)}^{t_{i+1}(s)} d\tau_i,$$

$$R(s)(-1)^j \int_{t_i(s)}^{t_{i+1}(s)} [D_{r-1}(\tau_i - \xi_{2N}(s)) - D_{r-1}(\tau_i - \xi_j(s)]\, d\tau_i,$$
$$j = 1, .., 2N-1,\quad i = 1, .., 2N,$$

ordered in the described way. Now unfolding $\det J(s)$ according to the first column, we get

$$\det J(s) = \sum_{i=1}^{2N} \left[(-1)^i \int_{t_i(s)}^{t_{i+1}(s)} P(s,\tau_i) d\tau_i \right] \det J_i(s), \tag{20}$$

where $J_i(s)$ are matrices with rows

$$1, \quad (-1)^j [D_{r-1}(t_1(s) - \xi_{2N}(s)) - D_{r-1}(t_1(s) - \xi_j(s)], \quad j = 1, .., 2N-1$$

$$R(s) \int_{t_k(s)}^{t_{k+1}(s)} d\tau_k, \quad R(s)(-1)^j \int_{t_k(s)}^{t_{k+1}(s)} [D_{r-1}(\tau_k - \xi_{2N}(s)) - D_{r-1}(\tau_k - \xi_j(s)] d\tau_k,$$
$$j = 1, .., 2N-1, k = 1, ..., 2N, k \neq i.$$

The determinants $J_i(0)$, $i = 1, .., 2N$ have the same sign. To calculate $J_i(0)$, we add all jth columns $j = 3, .., 2N$ to the second one. Then all elements of the second column will vanish except the first, which equals $P'(0, t_1(0))$. Thus

$$\det J_i(0) = -P'(0, t_1(0)) \det J'(0), \tag{21}$$

where $J_i'(s)$ are the matrices with rows

$$R(s) \int_{t_k(s)}^{t_{k+1}(s)} d\tau_k, \; R(s)(-1)^j \int_{t_k(s)}^{t_{k+1}(s)} [D_{r-1}(\tau_k - \xi_{2N}(s)) - D_{r-1}(\tau_k - \xi_j(s)] d\tau_k,$$
$$j = 2, .., 2N-1, k = 1, .., 2N, k \neq i, i = 1, .., 2N.$$

Denote $t\bar{a}u^i = (\tau_1, .., \tau_{i-1}, \tau_{i+1}, .., \tau_{2N})$. Then

(22) $\det J_i'(s) =$
$$(R(s))^{2N-1} \int_{t_2}^{t_3} .. \int_{t_{i-1}}^{t_i} \int_{t_{i+1}}^{t_{i+2}} .. \int_{t_{2N}}^{t_{2N+1}} \det J_i''(s, \bar{\tau}^i) d\tau_1 .. d\tau_{i-1} d\tau_{i+1} .. d\tau_{2N},$$

where $J_i''(s)$ are the matrices having rows

$$1 \; (-1)^j [D_{r-1}(\tau_k - \xi_{2N}(s)) - D_{r-1}(\tau_k - \xi_j(s)], \; j = 2, .., 2N-1, \; k = 1, .., 2N, \; k \neq i.$$

It follows by Lemma 3.2 that

$$\beta \det J_i''(0, \bar{zt}^i) \geq 0, \tag{23}$$

where $\beta = 1$ or -1.
We will show, that there is a point $(\tau_2^0, .., \tau_{2N}^0)$ at which

$$\det J_1''(0, \tau_2^0, .., \tau_{2N}^0) \neq 0. \tag{24}$$

Indeed by Rolle's theorem, the so-called interlacing conditions

$$t_i < \xi_i < t_{i+r-1}, \; i = 2,..,2N$$

hold for some cyclic arrangement of the points $\{t_i\}$ and $\{\xi_i\}$. The latter together with the inequalities $t_i \leq \tau_i \leq t_{i+1}$, $i = 2,..,2N$, shows the existence of points $(\tau_i^0,..,\tau_{2N}^0)$, such that

$$\tau_i^0 < \xi_i < t_{i+r-1} \leq \tau_{i+r-1}^0, \; i = 2,..,2N. \tag{25}$$

Then (24) follows from (25) and Lemma 3.2. We conclude from (20)–(24), that

$$\beta \det J(0) = \sum_{i=1}^{2N} |\int_{t_i^0}^{t_{i+1}^0} P_0(x)dx|| \det J_i(0)| > 0 \tag{26}$$

Therefore, by (19) and (26) we get

$$\det \Delta(0) \neq 0. \tag{27}$$

Hence, by the implicit function theorem, there exists a unique system of continuous functions

$$(R(s), c(s), \xi_1(s),.., \xi_{2N}(s), t_2(s),.., t_{2N}(s)) \; = \{f_k(s), k = 1,.., 4N+1\}$$

defined on a neighbourhood of 0, which satisfies (18).

Next following the idea in the proof of Theorem 3 from [1] one may show that the functions above are defined for each $s \in [0,1]$ and hence they constitute a solution for (18). The solution is unique for $s = 1$. We omit this part of the proof here and refer to [1] for details.

It remains to show the monotone dependence of R on e_i. There exists a unique $R = R(e_1,..,e_{2N})$ satisfying (18) for $s = 1$. Since $\det \Delta(1) \neq 0$, by the implicit function theorem, $R(e_1,..,e_{2N})$ is a differentiable function of $e_1,..,e_{2N}$. Moreover,

$$R_{e_i} = \frac{\det \Delta_i}{\det \Delta(1)},$$

where Δ_i differs from $\Delta(1)$ by its first column which has only one nonzero element, namely $(-1)^i e_i$, on position $i+1$. Using the same arguments as in the evaluation of $\Delta(s)$ we obtain

$$R_{e_i} = \frac{e_i |J_i(1)|}{\sum_{j=1}^{2N} e_j |J_j(1)|} > 0, \; i = 1,...,2N.$$

The theorem is proved.

Let us choose $e_i = |y_{i+1} - y_i|$, $i = 1, ..2N$. Integrating the unique periodic perfect spline $P(x)$ from Theorem 3.1, we obtain the following consequence.

Corollary 3.1. *Given N, r and $\{y_i\}_1^{2N}$ satisfying (1) there exists a unique periodic perfect spline P^* of degree r with $2N$ knots and a unique set of points $0 = t_1^* < ... < t_{2N}^* < 1$ for which*

1) $P^*(t_i^*) = y_i$, $i = 1, .., 2N$.

2) $P^{*'}(t_i^*) = 0$, $i = 1, .., 2N$.

Theorem 3.2. *The periodic perfect spline of Corollary 3.1 is the unique solution of (2) for $p = \infty$.*

We first prove the following auxiliary statement, which will be used in the proof of Theorem 3.2.

Lemma 3.3. *Given N, r and $\{y_i\}_1^{2N}$ satisfying (1) there is a unique $P^* \in \tilde{P}_{N,r}$ and points $0 = t_1^* < ... < t_{2N}^* < 1$ such that $P^*(t_i^*) = y_i$, $i = 1, .., 2N$, and*

(28)
$$\|P^{*(r)}\|_\infty < \|P^{(r)}\|_\infty$$

for all $P \in P_{N,r}$, $P \neq P^*$, satisfying the conditions $P(t_i) = y_i$, $i = 1, .., 2N$ for some points $\{t_i\}$.

Moreover, the extremal periodic spline P^* is characterized by the property

$$P^{*'}(t_i^*) = 0, \ i = 1, .., 2N.$$

Proof. It follows from Corollary 3.1 that there is a unique perfect spline $P^* \in P_{N,r}$ and a constant $R > 0$ such that

$$P^*(t_i^*) = y_i, \ P^{*'}(t_i^*) = 0, \ i = 1, .., 2N.$$

$$\|P^{*(r)}\|_\infty = R(e_1, .., e_{2N}), \text{ where } e_i = |y_{i+1} - y_i|, \ i = 1, .., 2N.$$

Now we suppose that P^* does not satisfy (28). Then there exists $P \in \tilde{P}_{N,r}$, such that $P \neq P^*$ and $P(t_i) = y_i$, for some points t_i, $i = 1, .., 2N$, which satisfy

$$\|P^{*(r)}\|_\infty \leq \|P^{(r)}\|_\infty.$$

But there exist points $t_i < s_i < t_{i+2}$, $i = 1, .., 2N$ for which

$$P'(s_i) = 0.$$

Let $P(s_i) = y'_i$, $i = 1,..2N$, $e'_i = |y'_{i+1} - y'_i|$, $i = 1,..,2N$. We have

$$e'_i \geq e_i, \ i = 1,..,2N$$

and there is a strict inequality at least for one i. But R is strictly increasing. Therefore $R(e_1,..,e_{2N}) < R(e'_1,..,e'_{2N})$, which is a contradiction. The proof is completed. ■

Proof of Theorem 3.2. Let P^* be the extremal periodic perfect spline of Lemma 3.3 with $P^*(t^*_i) = y_i$ and $P^{*'}(t^*_i) = 0$, $i = 1,..,2N$. We will show, that P^* is a unique solution of the problem (2) for $p = \infty$. Assume that $f \in \tilde{W}^r_\infty(t;y)$, for some $t \in \Sigma_{2N}$. There exists a periodic perfect spline P of degree r with $2N$ knots for which $P(t_i) = f(t_i) = y_i$, $i = 1,..,2N$ and $\|P^{(r)}\|_\infty \leq \|f^{(r)}\|_\infty$. This fact is proved in [4]. By Lemma 3.3 it follows that if $t \neq t^*$, $\|P^{*(r)}_\infty\| \leq \|P^{(r)}\|_\infty$. Therefore any solution f of the problem (2) is from $\tilde{W}^r_\infty(t^*;e)$ and $\|f^{(r)}\| = \|P^{*(r)}\|_\infty$.

Next we shall show that $f'(t^*_i) = 0$, $i = 1,..,2N$ for any f as above. Assume that $f'(t^*_i) \neq 0$ for some $j \in \{1,..,2N\}$. It follows by the character of the data that there exists $s_j \in (t^*_{j-1}, t^*_{j+1})$ such that $f'(s_j) = 0$. Let P be a periodic perfect spline of degree r with $2N$ knots for which $P(t^*_i) = y_i$, $i = 1,..,2N$, $i \neq j$, and $P(s_j) = f(s_j)$. Then P attains the value y_i at least twice in (t^*_{j-1}, t^*_{j+1}) and $P \not\equiv P^*$. Thus $\|P^{*(r)}\|_\infty \leq \|f^{(r)}\|_\infty$ and from Lemma 3.3 $\|P^{*(r)}\|_\infty < \|P^{(r)}\|_\infty$. This contradicts the minimality property of f. Thus $f'(t^*_i) = 0$, $i = 1,..,2N$.

Assume that $f \not\equiv P^*$. Then $f \not\equiv P^*$ on (t^*_j, t^*_{j+1}) for some $j \in 1,..,2N$. Since $(P^* - f)(t^*_i) = 0$, $i = j, j+1$, $(P^* - f)'(x)$ must change sign on (t^*_j, t^*_{j+1}). Thus for $\epsilon > 0$, sufficiently small, $(P^* - (1-\epsilon)f)'(x)$ has a sign change in (t^*_j, t^*_{j+1}). Moreover $(P^* - (1-\epsilon)f)'(t^*_i) = 0$, $i = 1,..,2N$ and hence the function $(P^* - (1-\epsilon)f)'(x)$ has at least $(2N+1)$ distinct zeros in $[0,1)$. Then, by Rolle's theorem $(P^* - (1-\epsilon)f)^{(r)}(x)$ has at least $(2N+1)$ sign changes on $[0,1)$. But $P^{*(r)}(x)$, and consequently $(P^* - (1-\epsilon)f)^{(r)}(x)$, has exactly $2N$ sign changes, a contradiction. The proof is completed. ■

4. Characterization of the solution to (2) for $p \in (1,\infty]$, $r = 2$

First we consider the case $p \in (1,\infty)$. By Theorem 2.1 there exists a solution of (2), and each solution (t^*, f^*), $t \in \Sigma_{2N}$, $f^* \in \tilde{W}^2_p[0,1]$ satisfies

$$f^{*''}(x) = |\sum_{i=1}^{2N} \alpha_i \tilde{B}_{i,2}(x)|^{q-1} \operatorname{sign}(\sum_{i=1}^{2N} \alpha_i \tilde{B}_{i,2}(x)), \tag{29}$$

where $1/p + 1/q = 1$ and $\tilde{B}_{i,2}(x)$ is the periodic B-spline of first degree with knots $t_i^*, t_{i+1}^*, t_{i+2}^*$ and

(30) $$f^{*'}(t_i^*) = 0,\ i = 1,..2N.$$

Denote $\Delta y_i = y_{i+1} - y_i$. For $t \in \Sigma_{2N}$, set $h_i = t_{i+1} - t_i$, $i = 1,..,2N$.

Theorem 4.1. *Given N and $y_1,..,y_{2N}$ satisfying (1) and $p \in (1,\infty)$, $r = 2$ there exists a unique solution (t^*, f_p^*) of (2). Moreover, the extremal nodes t^* are defined by*

(31) $$t_1^* = 0,\ t_{i+1}^* = t_i^* + h_i^*,\ i = 1,...,2N,$$

where

(32) $$h_i^* = \frac{|\Delta y_i|^{1/2}}{\sum_{i=1}^{2N} |\Delta y_i|^{1/2}},\ i = 1,...2N,$$

and the smoothest periodic interpolant is given by the expression

(33) $$f_p^*(x) = (2^q/q)(\Delta y_i/h_i^{*q+1})|-h_i^*/2 + x - t_i^*|\mathrm{sign}(-h_i^*/2 + x - t_i^*)$$
$$+(q+1)\Delta y_i(x - t_i^*)/h_i^* + y_i - \Delta y_i/(2q),$$

for $x \in [t_i^*, t_{i+1}^*]$, $i = 1,..,2N$, $(t_{2N+1}^* = t_1^* + 1)$.

Proof. Let (t^*, f^*) be a solution of (2). We set $p_i(x) = f^*(x)$ for $x \in [t_i^*, t_{i+1}^*]$, $i = 1,..,2N$. From the interpolation conditions and Theorem 2.1 we have

(34) $$p_i(t_i^*) = y_i,\ p_i(t_i + 1^*) = y_{i+1},\ p_i'(t_i^*) = 0,\ p_i'(t_{i+1}^*) = 0,$$
$$p_i''(x) = |\gamma_i + \beta_i(x - t_i^*)|^{q-1}\mathrm{sign}(\gamma_i + \beta_i(x - t_i^*)),\ x \in [t_i^*, t_{i+1}^*],\ \beta_i \neq 0.$$

It is not difficult to obtain from (34) the following relations

(35) $$|\gamma_i + \beta_i h_i^*| = |\gamma_i|,\ -h_i^*/2 = \gamma_i/\beta_i,$$

(36) $$-|\gamma_i|^{q-1}h_i^{*2}\mathrm{sign}(\gamma_i)/[2(q+1)] = \Delta y_i.$$

Using the continuity of $f^{*''}$ at the knots t_i^*, $i = 1,..,2N$, gives

$$p_i''(t_{i+1}^*) = p_{i+1}''(t_{i+1}^*),\ i = 1,..,2N\ (p_{2N+1}(x) = p_1(x))$$

By the latter we get

(37) $$2(q+1)\Delta y_i/h_i^{*2} = -2(q+1)\Delta y_{i+1}/h_{i+1}^{*2},\ i = 1,..,2N.$$

Then

(38) $$h_{i+1}^{*2}/h_i^{*2} = -\Delta y_{i+1}/\Delta y_i,\ i = 1,..,2N-1.$$

In view of the equality $\sum_{i=1}^{2N} h_i^* = 1$ and (38) we obtain the relations (31), (32) which define t^* uniquely.

Using (34)–(37) it is easy to show that $f^*(x)$ has the form given in (33). The proof is completed. ■

Now, letting p tend to ∞ we obtain the solution of (2) for the case $p = \infty$, $r = 2$.

Theorem 4.2. *Given N and $y_1, .., y_{2N}$ satisfying (1) and $p = \infty$, $r = 2$ there exists a unique solution (t^*, f_∞^*) of (2). Moreover, the extremal nodes t^* are defined by the formulae (31), (32) and the smoothest periodic interpolant by*

$$(39) \qquad f_\infty^*(x) = 2\Delta y_i/h_i^{*2}| - h_i^*/2 + x - t_i^*|^2 \mathrm{sign}(-h_i^*/2 + x - t_i^*)$$

$$+2\Delta y(x - t_i^*)/h_i^* + y_i - \Delta y_i/2,$$

for $x \in [t_i^*, t_{i+1}^*]$, $i = 1, .., 2N$, $(t_{2N+1}^* = t_1^* + 1)$.

Proof. Let $p \in (1, \infty)$, $1/p + 1/q = 1$ and f_p^* be the unique smoothest periodic interpolant from $\tilde{W}_p^2[0,1]$. From (33),

$$f_p^{*''}(x) = (q+1)\Delta y_i[(2/h_i^*)^{q+1}/2]| - h_i^*/2 + x - t_i^*|\mathrm{sign}(-h_i^*/2 + x - t_i^*)$$

for $x \in [t_i^*, t_{i+1}^*]$, $i = 1, .., 2N$, where the nodes t_i^* are defined by the formulae (31), (32) and do not depend on the parameter p. Denote by f_∞^* the function

$$f_\infty^*(x) = \lim_{p \to \infty} f_p^*(x), \; x \in [0,1].$$

Then,

$$f_\infty^{*''}(x) = \Delta y_i(2/h_i^*)^2 \mathrm{sign}(-h_i^*/2 + x - t_i^*), \; x \in [t_i^*, t_{i+1}^*], \; i = 1, .., 2N.$$

From the above form of $f_\infty^{*''}$ and (31) it follows that f_∞^* is a perfect periodic spline of degree 2 with knots $(t_i^* + t_{i+1}^*)/2$, $i = 1, .., 2N$. Moreover, $f_\infty^*(t_i^*) = y_i$ and $f_\infty^{*'}(t_i^*) = 0$, $i = 1, .., 2N$. Thus, t^*, f_∞^* satisfy the characterization of the solution of (2) given by Theorem 3.2 and since the solution in $\tilde{W}_\infty^2[0,1]$ is unique, the proof is completed. ■

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