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On the R-Order of Convergence of Classes of Iterative Methods

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In this note we consider classes of interval methods. Improved estimates for their R-order of convergence are derived.

1. Introduction

Recently many authors have considered two-sided and interval methods which guarantee a possibility for practical solution of different problems in numerical analysis. A machine realization of the methods is proposed in the spirit of the computing conception, which assumes the utilization of a computer which executes the arithmetic operations with directed rounding in the sense of [1].

Parallel interval iteration can be found in [3] (see also [18]). The method makes use of advanced computer arithmetic and has been recently realized in the frames of two program systems, which provide such arithmetic: PASCAL-SC [19] and HIFICOMP [20].

The following relations appear for a class of single-step iterative methods for the simultaneous determination of the zeros of n-degree polynomial (for instance, the methods considered in Alefeld and Herzberger [2], Petkovic [3], [17])

(1) $$h_i^{(m+1)} \leq \frac{1}{n-1}\left(h_i^{(m)}\right)^p \left(\sum_{j<i} h_j^{(m+1)} + \sum_{j>i}\left(h_j^{(m)}\right)^q\right),$$

$i = 1, \dots, n; \quad p, q \in N.$

We shall assume that the starting approximations $z_1^{(0)}, \dots, z_n^{(0)}$ (where $z_i^{(0)}$ can be either the points or circles in interval arithmetic) are chosen sufficiently close to the zeros $\xi_1, \dots, \xi_n$ so that (in the case when $z_i^{(j)}$ are points)

$$h_i^{(0)} \leq h = \max_{1\leq i\leq n} h_i^{(0)} < 1, \quad h_i^{(j)} = |z_i^{(j)} - \xi_i|, \; j = 0, 1, \dots, \; i = 1, 2, \dots, n.$$

Theorem A. (M. Petkovic, L. Petkovic, L. Stefanovic [4]). *The R-order of convergence (see J. Ortega, W. Rheinboldt [5]) of a simultaneous iterative process I for which (1) is valid, is bounded from below by*

(2) $$O_R(I, \xi) > p + q + \frac{pq}{(n-1)(p+q)} = \alpha(n, p, q), \quad \xi = (\xi_1, \xi_2, \dots \xi_n).$$

Let an iterative method I in a Banach space B produce sequences of iterates $\{x^{(k)}\}$ with $\lim_{k\to\infty} x^{(k)} = x^*$. In many cases, one can show for the corresponding sequences of errors $e^{(k)} = ||x^{(k)} - x^*||$ the recursion

$$e^{(k+11)} \leq \gamma \prod_{i=0}^{n}\left(e^{(k-i)}\right)^{q^i(p+1)}, \quad k \geq 0,$$

where γ, p, q are positive and independent of k. In order to calculate the R-order of convergence $O_R(I, x^*)$ of I one has to compute the unique positive root $\sigma_{p,q}^{(n+1)}$ of the polynomial

$$P_{n+1}(x) = x^{(n+1)} - (p+1)\sum_{i=0}^{n} q^i x^{n-i}, \quad p \geq 0, q > 0.$$

J. Schmidt [7] has shown that

(3) $$O_R(I, x^*) \geq \sigma_{p,q}^{(n+1)}$$

is valid. The problem of determination of bounds for $\sigma_p, q^{(n)}$ is considered in [8–12, 21, 23].

2. Main Results

The purpose of this paper is to give better lower estimates in (2) and (3). Using the same notation as in theorem A, we state the following

Theorem.

$$O_R(I,\xi) > 3p + 2q + -(2p+q)\left(\frac{p+q}{2p+q}\right)^{\frac{q}{(n-1)(p+q)}} \tag{4}$$

P r o o f. More readily applied is the following theorem (see E. D e u t s c h [6]): Let $A = (a_{i,j})$ be a nonnegative and irreducible $n \times n$ matrix and let the positive vectors x, y be defined by $Ax = Dx$, $A^T y = Dy$, where $D = \operatorname{diag}(d_1, \ldots, d_n) > 0$. If x is not an eigenvector of A then it follows for the spectral radius $\rho(A)$ of A

$$\rho(A) > \frac{y^T Dx}{y^T x} \ . \tag{5}$$

In [3], M. P e t k o v i c derive the estimation (2) by means of (5). We shall use the improve estimate from E. D e u t s c h 's theorem [6] which states that for all

$$t > \rho(A) + \max_{1 \le i \le n} \{d_i - a_{ii}\}$$

it is fulfilled

$$\rho(A) > G = t - \prod_{i=1}^{n} (t - d_i)^{\frac{x_i y_i}{y^T x}} > \frac{y^T Dx}{y^T x} \ . \tag{6}$$

The matrix

$$A_n(p,q) = \begin{bmatrix} p & q & & & 0 \\ & p & q & & \\ 0 & & \ddots & \ddots & \\ & & & p & q \\ p & q & \ldots & 0 & p \end{bmatrix} .$$

corresponds to the recursion (1) (see M. P e t k o v i c [3]) and the estimation (6) will be applied to $A_n(p,q)$. It follows from (1) that

$$h_i^{(m+1)} \le h^{s_i^{(m+1)}} ,$$

where the vectors $s^{(m)} = [s_1^{(m)}, \ldots, s_n^{(m)}]^T$ are successively computed by

$$s^{(m+1)} = A_n(p,q)\, s^{(m)}$$

starting with $s^{(0)} = [1, \ldots, 1]^T$. Therefore, we have

$$\begin{aligned} D &= \operatorname{diag}(p+q, p+q, \ldots p+q, 2p+q), \\ y &= (\gamma/q)\,[p \;\; p+q \; \ldots \; p+q \;\; q]^T, \qquad \gamma > 0, \\ y^T x &= (\gamma/q)\,(p+q)\,(n-1), \\ y^T D x &= (\gamma/q)\,(\,(p+q)\,(n-1) + pq). \end{aligned}$$

In view of (6) and since

$$t = \max_i d_i + \max_i (d_i - a_{ii}) = 3p + 2q$$

it follows that

$$\begin{aligned} O_R(I, \xi) &\geq \rho(A_n(p,q)) > t - \prod_{i=1}^{n} (t - d_i)^{x_i y_i / y^T x} \\ &= 3p + 2q - \sqrt[(n-1)(p+q)]{(2p+q)^{(n-1)(p+q)} \left(\frac{p+q}{2p+q} \right)^q}\,. \end{aligned}$$

This is an improvement of the bound in theorem A. Some numerical comparisons between the estimations (2) and (4) are given in the next table.

n	p	q	$\alpha(n,p,q)$	$\beta(n,p,q)$
2	1	1	2.5000...	2.5505...
2	3	3	7.5000...	7.6515...
2	5	5	12.5000...	12.7526...
5	1	1	2.1250...	2.1483...
5	3	3	6.3750...	6.4448...
5	5	5	10.6250...	10.7413...
10	1	1	2.0556...	2.0668...
10	3	3	6.1667...	6.2005...
10	5	5	10.2778...	10.3341...

Using the same notation as in J. Herzberger's theorem [8] the next inequalities gives an estimation for $\sigma_{p,q}^{(n)}$ in (3) (see [22]):

$$(7) \quad p+q+1 - A_{p,q}^{(n)} \frac{(p+1)\,q^n}{(p+q+1)^n} < \sigma_{p,q}^{(n)} < p+q+1 - B_{p,q}^{(n)} \frac{(p+1)q^n}{(p+q+1)^n}\,,$$

where

$$A_{p,q}^{(n)} = \frac{2}{1 + \sqrt{1 - 4n(p+1)q^n/(p+q+1)^{(n+1)}}}, \quad B_{p,q}^{(n)} = (1 + \epsilon_{p,q}^{(n)})^n,$$

$$\epsilon_{p,q}^{(n)} = \frac{2(p+1)q^n \left((p+q+1)^{n+1} - (p+1)q^n (n+1)\right)^{-1}}{1 + \sqrt{1 - 4n(p+1)^2 q^{2n} / \left((p+q+1)^{n+1} - (p+1)q^n(n+1)\right)^2}}.$$

Using the estimation (7) we may establish the inequalities

$$O_R(0, \{e^{(k)}\}) > \sigma_{p,q}^{(n+1)} > p + q + 1 - A_{p,q}^{(n+1)} \frac{(p+1)q^{n+1}}{(p+q+1)^{n+1}},$$

which slightly improves the estimations given in [8].

The basic facts about R-order of convergence of sequences (including interval ones) are given in W. Burmeister, J. Schmidt [13], J. Herzberger [14], N. Kjurkchiev [15].

3. Applications

Following J. Herzberger [8], as an application of our theorem we consider a class of iteration methods for the successive improvement of an including interval matrix $X^{(0)}$ for the inverse of a nonsingular matrix A

$$\begin{aligned}
Y^{(0)} &= X^{(0)}, \\
Y^{(k+1)} &= \{m(Y^{(k)}) \sum_{i=0}^{r-2} (I - Am(Y^{(k)}))^i + \\
&\quad + X^{(k)}(I - Am(Y^{(k)}))^{r-1}\} \bigcap X^{(k)}, \\
X^{(k+1)} &= \{m(Y^{(k+1)}) \sum_{i=0}^{r-2} (I - Am(Y^{(k+1)}))^i + \\
&\quad + Y^{(k+1)}(I - Am(Y^{(k+1)}))^{r-1}\} \bigcap Y^{(k+1)},
\end{aligned} \tag{8}$$

(the parameter $r \in N$ has to be greater than 1 and $m(X)$ is the midpoint matrix of X). For the R-order of convergence of procedure (8) instead of the estimation

$$O_R((8), A^{-1}) > \frac{n+1}{n+2}(2r - 1)$$

given in [8], we get from (7)

$$O_R(\,(8), A^{-1}) > 2r - 1 - \frac{r(r-1)^{n+1}}{(2r-1)^{n+1}}\, A^{(n+1)}_{r-1,r-1}\,.$$

Consider the interval iteration process of Newton type for finding real zero ξ of nonlinear equation $f(x) = 0$ (see J. Herzberger [16])

$$\begin{aligned} Y^{(0)} &= X^{(0)}, \\ Y^{(k+1)} &= \left\{ m(X^{(k)}) - \frac{f(m(X^{(k)}))}{f'(Y^{(k)})} \right\} \bigcap X^{(k)}, \\ X^{(k+1)} &= \left\{ m(Y^{(k+1)}) - \frac{f(m(Y^{(k+1)}))}{f'(Y^{(k+1)})} \right\} \bigcap Y^{(k+1)}. \end{aligned} \tag{9}$$

Using the estimation (7) we may establish

$$O((9), \xi) > 3 - \frac{2}{3^{n+1}}\, A^{(n+1)}_{1,1}\,.$$

These results can be applied also for determination of the computational efficiency of the considered iterative methods.

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