Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica
Balkanica

New Series Vol.8, 1994, Fasc.4

Improvements of Cardinal Inequalities for Topological Spaces and k-structures

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The aim of this paper is to improve several results about the restriction of the cardinality of a given topological space done by U. N. B. Dissanayake and S. Willard [3], Sun Shu-Hao [9] and others. A combinatorial theorem about the recently defined by R. E. Hodel [5, 6] k-structures is also obtained. As corollaries we get some old and new cardinal inequalities for topological and neighborhood (in the sense of [10]) spaces.

1. Introduction

All spaces are assumed to be Hausdorff and standard notations following [2], [4] and [7] are used.

In [3] the following cardinal invariant for a topological space X is defined — $aL(X) = \omega \cdot \min\{\tau$: for every family $\gamma \subseteq \exp X$ consisting of open sets and every closed $F \subseteq X$ such that $\bigcup\gamma \supseteq F$ there is $\gamma' \in [\gamma]^{\leq\tau}$ for which $\bigcup\{\overline{U} : U \in \gamma'\} \supseteq F\}$, it was noticed that $wL(X) \leq aL(X) \leq L(X)$ and $aL(X) = L(X)$ for regular X. It was also proved that for a Hausdorff space X we have that $|X| \leq \exp aL(X)\cdot\pi\chi(X)\cdot\psi_c(X)\cdot t(X)$, where $\psi_c(X) = \omega\cdot\min\{\tau$: for each $x \in X$ there is a family of open neighborhoods $\{U_\alpha(x) : \alpha \in \tau\}$ of x such that $\{x\} = \bigcap\{\overline{U_\alpha(x)} : \alpha \in \tau\}\}$.

On the other hand, Sun Shu-Hao introduced in [9] the invariant $kL(X) = \omega\cdot\min\{\tau$: there is an $A \subseteq X$, $|A| \leq 2^\tau$ such that $(\star)$: for each open cover γ of X there are $\gamma' \in [\gamma]^{\leq\tau}$ and $B \in [A]^{\leq\tau}$ such that $X = \bigcup\gamma' \cup \overline{B}\ \}$ and observed that $kL(X) \leq \min\{d(X), L(X), s(X)\}$. ~~He also~~ proved that for a Hausdorff space X we have $|X| \leq \exp kL(X) \cdot \psi_c(X) \cdot t(X)$. To do this he used the following:

Lemma 1. *If X is a topological Hausdorff space, $L \in [X]^{\leq \exp \tau}$ and $\psi_c(X) \cdot t(X) \leq \tau$ then $|\overline{L}| \leq 2^\tau$.*

Here the following more general cardinal function is defined:

$kaL(X) = \omega \cdot \min\{\tau$: there is an $A \subseteq X, |A| \leq 2^\tau$ such that $(\star\star)$: if $\mathcal{W}$ is a family of open sets, $F \subseteq X$ is closed and $F \subseteq \bigcup \mathcal{W}$ then there are $\mathcal{W}' \in [\mathcal{W}]^{\leq \tau}$ and $B \in [A]^{\leq \tau}$ such that $F \subseteq \bigcup\{\overline{U} : U \in \mathcal{W}'\} \cup \overline{B}\}$.

Let us note that $kwL(X) \leq kqL(X) \leq kaL(X) \leq kL(X)$ (see unknown definitions in [12]).

2. Main theorem

Theorem 1. *If X is a Hausdorff topological space then:*

$$|X| \leq \exp kaL(X) \cdot \psi_c(X) \cdot t(X).$$

Proof. Let $kaL(X) \cdot \psi_c(X) \cdot t(X) \leq \tau$. Let $A \subseteq X, |A| \leq 2^\tau$ be with the property $(\star\star)$ and for every $x \in X$ let us fix a family of neighborhoods of the point x — $\mathcal{W}(x), |\mathcal{W}(x)| \leq \tau$ such that $\{x\} = \bigcap\{\overline{U} : U \in \mathcal{W}(x)\}$. By transfinite induction we shall define two families — $\{H_\alpha : \alpha \in \tau^+\}$ and $\{\mathcal{B}_\alpha : \alpha \in \tau^+\}$ such that:

1) H_α is a closed subspace of X and $|H_\alpha| \leq 2^\tau$ for every $\alpha \in \tau^+$.

2) $H_\alpha \subseteq H_{\alpha'}$ if $\alpha \leq \alpha' \in \tau^+$.

3) If $\alpha \in \tau^+$ and $\{H_\beta : \beta \in \alpha\}$ are already defined then $\mathcal{B}_\alpha = \bigcup\{\mathcal{W}(x) : x \in \bigcup\{H_\beta : \beta \in \alpha\}\}$.

4) If $\mathcal{W} \in [\mathcal{B}_\alpha]^{\leq \tau}$, $B \in [A]^{\leq \tau}$ and $X \setminus \{\bigcup\{\overline{U} : U \in \mathcal{W}\} \cup \overline{B}) \neq \phi$ then $H_\alpha \setminus (\bigcup\{\overline{U} : U \in \mathcal{W}\} \cup \overline{B}) \neq \phi$.

Let $\alpha \in \tau^+$ and $\{H_\beta : \beta \in \alpha\}$ and $\{\mathcal{B}_\beta : \beta \in \alpha\}$ be already defined with properties 1) – 4).

Let $E_\alpha = \{(\mathcal{W}, B) : \mathcal{W} \in [\mathcal{B}_\alpha]^{\leq \tau}, B \in [A]^{\leq \tau}$ and $X \setminus (\bigcup\{\overline{U} : U \in \mathcal{W}\} \cup \overline{B}) \neq \phi\}$. For every pair $(\mathcal{W}, B) \in E_\alpha$ we choose a point $\Phi(\mathcal{W}, B) \in X \setminus (\bigcup\{\overline{U} : U \in \mathcal{W}\} \cup \overline{B}) \neq \phi$ and let $C_\alpha = \{\Phi(\mathcal{W}, B) : (\mathcal{W}, B) \in E_\alpha\}$. Since $|E_\alpha| \leq 2^\tau$ we have that $|C_\alpha| \leq 2^\tau$. Finally, we put $H_\alpha = \overline{C_\alpha \cup \bigcup\{H_\beta : \beta \in \alpha\}}$. Since $|C_\alpha \cup \bigcup\{H_\beta : \beta \in \alpha\}| \leq 2^\tau$ and $\psi_c(X) \cdot t(X) \leq \tau$, we obtain using Lemma 1 that $|H_\alpha| \leq 2^\tau$. It can be easily seen that the conditions 1) – 4) are satisfied.

Let $H = \bigcup\{H_\alpha : \alpha \in \tau^+\}$. We shall show that H is τ-closed, i. e. if $M \subseteq [H]^{\leq\tau}$ then $\overline{M} \subseteq H$. Since τ^+ is a regular cardinal number and 2) is satisfied, then for every $M \in [H]^{\leq\tau}$ there is an $\alpha \in \tau^+$ such that $M \subseteq H_\alpha$. Then $\overline{M} \subseteq \overline{H}_\alpha = H_\alpha \subseteq H$. From $t(X) \leq \tau$ we conclude that H is also closed.

Let us show that $X = H \cup \overline{A}$ i. e. $X \backslash H \subseteq \overline{A}$. Let $q \in X \backslash H$ and let us show that $q \in \overline{A}$. For every $p \in H$ we can choose $V_p \in \mathcal{W}(p)$ such that $q \notin \overline{V_p}$. Let $\mu = \{V_p : p \in H\}$. We have that $\bigcup \mu \supseteq H$ and from $kaL(X) \leq \tau$ we can choose $\mu_0 \in [\mu]^{\leq\tau}$ and $B \in [A]^{\leq\tau}$ such that $H \subseteq \bigcup\{\overline{U} : U \in \mu_0\} \cup \overline{B}$. If $q \in \overline{B} \subseteq \overline{A}$ we are done. Let $q \notin \overline{B}$. So $X \backslash (\bigcup\{\overline{U} : U \in \mu_0\} \cup \overline{B}) \neq \phi$. From the other hand $\mu_0 \subseteq \mu = \{V_p : p \in H\} \subseteq \bigcup\{\mathcal{W}(p) : p \in H\} = \bigcup\{\mathcal{B}_\alpha : \alpha \in \tau^+\}$ and we have that $\mathcal{B}_\alpha \subseteq \mathcal{B}_{\alpha'}$ if $\alpha \leq \alpha' \in \tau^+$. From the regularity of τ^+ and the fact that $|\mu_0| \leq \tau$ there is an $\alpha_0 \in \tau^+$ such that $\mu_0 \subseteq \mathcal{B}_{\alpha_0}$. Then we have already chosen a point $\Phi(\mu_0, B) \in (X \backslash (\bigcup\{\overline{U} : U \in \mu_0\} \cup \overline{B})) \bigcap H_{\alpha_0} \subseteq H$ and in the same time $\bigcup\{\overline{U} : U \in \mu_0\} \cup \overline{B} \supseteq H$ — a contradiction.

Corollary 1 [8]. *For every regular topological space X we have that* $|X| \leq \exp kL(X) \cdot \chi(X)$.

Corollary 2 [9]. *For every Hausdorff topological space X we have that* $|X| \leq \exp kL(X) \cdot \psi_c(X) \cdot t(X)$.

Corollary 3. *For every Hausdorff topological space X we have that* $|X| \leq \exp aL(X) \cdot \psi_c(X) \cdot t(X)$.

Corollary 4 [3]. *For every Hausdorff topological space X we have that* $|X| \leq \exp aL(X) \cdot \psi_c(X) \cdot t(X) \cdot \pi\chi(X)$.

Corollary 5 [1]. *For every Hausdorff topological space X we have that* $|X| \leq \exp L(X) \cdot \chi(X)$.

3. Results about k-structures

In [5,6], R. E. Hodel introduced and studied the notion of k-structure that generalizes the notion of a topological and closure space in the sense of E. Cech [2]. He considered some conditions that correspond to certain cardinal functions for the topological case and proved some combinatorial theorems from which several well known topological cardinal inequalities hold. Let us remind some of the definitions.

Let E be a set, k — an infinite cardinal number. A k-structure on E is the collection $\mathcal{V} = \{V(p, \gamma) : p \in E, \gamma \in k\}$ of subsets of E such that

$p \in \bigcap\{V(p,\gamma) : \gamma \in k\}$. For each $A \subseteq E$ let $A^* = \{p \in E : V(p,\gamma) \cap A \neq \phi$ for each $\gamma \in k\}$. We write $(E,\mathcal{V})$ to denote the fact that E is a set with a k-structure $\mathcal{V} = \{V(p,\gamma) : p \in E, \gamma \in k\}$. Let us consider the following conditions on a k-structure $(E,\mathcal{V})$ as defined in [6]:

S) $\bigcap\{V(p,\gamma) : \gamma \in k\} = \{p\}$ for all $p \in E$.

H) If $p \neq q$ there are $\alpha, \beta \in k$ such that $V(p,\alpha) \cap V(q,\beta) = \phi$.

I) Given $p \in E$ and $\alpha, \beta \in k$ there exists $\gamma \in k$ such that $V(p,\gamma) \subseteq V(p,\alpha) \cap V(p,\beta)$.

The subset $A \subseteq E$ is called $*$-closed if $A^* = A$.

Here the conditions KAL) and S*) will be introduced and studied.

S*) $\bigcap\{V(p,\gamma)^* : \gamma \in k\} = \{p\}$ for all $p \in E$.

KAL) There is $A \subseteq E$ with $|A| \leq 2^k$ such that for every $L \subseteq E$ with $L^* = L$ and every family $\mathcal{V}_0 \subseteq \mathcal{V}$ that covers L, a subfamily $\mathcal{W} \in [\mathcal{V}_0]^{\leq k}$ and $F \in [A]^{\leq k}$ could be found such that $L \subseteq \bigcup\{W^* : W \in \mathcal{W}\} \cup F^*$.

We have that H) $\Rightarrow$ S*) $\Rightarrow$ S) and C) $\Rightarrow$ KL) $\Rightarrow$ KAL), where KL) and C) are the k-structure notions that correspond to topological properties $kL(X)$ and $L(X)$.

Theorem 2. *Let $(E,\mathcal{V})$ be a k-structure that satisfies S*) + KAL) + If $B \in [E]^{\leq k}$ then $B^* \in [E]^{\leq \exp k}$). Then $|E| \leq 2^k$.*

Proof. Let $A \in E$ be such as in the property KAL). Construct a sequence $\{E_\alpha : \alpha \in k^+\}$ of subsets of k such that the following conditions hold for every $\alpha' \in \alpha \in k^+$:

(1) $|E_\alpha| \leq 2^k$, $E_{\alpha'} \subseteq E_\alpha$.

(2) If $\alpha \in \tau^+$ and $\{E_\beta : \beta \in \alpha\}$ are already defined then $\mathcal{B}_\alpha = \{V(x,\gamma) : x \in \bigcup\{E_\beta : \beta \in \alpha\}, \gamma \in k\}$.

(3) If $N \subseteq \bigcup\{E_\beta : \beta \in \alpha\}$ and $|N| \leq k$ then $N^* \subseteq E_\alpha$.

(4) If $\mathcal{W} \in [\mathcal{B}_\alpha]^{\leq k}$, $B \in [A]^{\leq k}$ and $E\backslash(\bigcup\{W^* : W \in \mathcal{W}\} \cup B^*) \neq \phi$ then $E_\alpha\backslash(\bigcup\{W^* : W \in \mathcal{W}\} \cup B^*) \neq \phi$.

Let $\alpha \in k^+$ and $\{E_\beta : \beta \in \alpha\}$ be already defined by the properties (1) – (4). We have that $|\mathcal{B}_\alpha| \leq 2^k$. When possible let us choose a point $p_{\mathcal{W},B} \in E\backslash(\bigcup\{W^* : W \in \mathcal{W}\} \cup B^*) \neq \phi$, where $B \in [A]^{\leq k}$ and $\mathcal{W} \in [\mathcal{B}_\alpha]^{\leq k}$. Let M_α be the set of all those $p_{\mathcal{W},B}$. Since $|A| \leq 2^k$ and $|B| \leq k$, then $|M_\alpha| \leq 2^k$.

Let $E'_\alpha = \bigcup\{E_\beta : \beta \in \alpha\} \cup M_\alpha$. Then $|E'_\alpha| \leq 2^k$. Finally, we put $E_\alpha = \bigcup\{B^* : B \in [E'_\alpha]^{\leq k}\}$. We have that $|E_\alpha| \leq 2^k$.

Let $L = \bigcup\{E_\alpha : \alpha \in k^+\}$. We shall prove that $L^* = L$. Let $p \in L^*$. Then for each $\gamma \in k$ there exists $x_\gamma \in V(p,\gamma) \cap L$. Let $N = \{x_\gamma : \gamma \in k\}$ and let us also note that $p \in N^*$. Since k^+ is regular and $N \subseteq L$, there exists $\alpha \in k^+$ such that $N \in [\bigcup\{E_\beta : \beta \in \alpha\}]^{\leq k}$. Then $N^* \subseteq E_\alpha \subseteq L$ and therefore $p \in L$.

We shall prove that $E = A^* \cup L$ i.e. $E \backslash L \subseteq A^*$. Let us fix $q \in E \backslash L$. Then by S*), for every $l \in L$ there is an $\alpha(l,q) \in k$ such that $q \notin V(l, \alpha(l,q))^*$. Let $\mathcal{W} = \{V(l, \alpha(l,q)) : l \in L\}$. We have that $\mathcal{W} \subseteq \mathcal{V}$ and $\bigcup \mathcal{W} \supseteq L$. From KAL) there are $\mathcal{U} \in [\mathcal{W}]^{\leq k}$ and $F \in [A]^{\leq k}$ such that $L \subseteq \bigcup\{U^* : U \in \mathcal{U}\} \cup F^*$. If $q \in F^* \subseteq A^*$ we are done. If $q \notin F^*$ then $q \notin \bigcup\{U^* : U \in \mathcal{U}\} \cup F^*$. Therefore $E \backslash (\bigcup\{U^* : U \in \mathcal{U}\} \cup F^*) \neq \phi$. We also have that $\mathcal{U} \in [\mathcal{B}_\alpha]^{\leq k}$ for some $\alpha \in k^+$. This means that we have already chosen a point $p_{\mathcal{U},F} \in (E \backslash (\bigcup\{U^* : U \in \mathcal{U}\} \cup F^*)) \cap E_\alpha$ and $p_{\mathcal{U},F} \in E_\alpha \subseteq L \subseteq \bigcup\{U^* : U \in \mathcal{U}\} \cup F^*$. This contradiction completes the proof.

Corollary 6 [13]. *Let $(E, \mathcal{V})$ be a k-structure that satisfies S*) + KL) + If $B \in [E]^{\leq k}$ then $B^* \in [E]^{\leq \exp k}$). Then $|E| \leq 2^k$.*

Corollary 7 [13]. *Let $(E, \mathcal{V})$ be a k-structure satisfying H) + I) + KL). Then $|E| \leq 2^k$.*

Proof. It follows from H) + I) and Theorem P in [6] that if $A \in [E]^{\leq k}$ then $A^* \in [E]^{\leq \exp k}$. Therefore the conditions of Corollary 6 are fulfilled.

Corollary 8 [6]. *Let $(E, \mathcal{V})$ be a k-structure satisfying S*) + C) + If $B \in [E]^{\leq k}$ then $B^* \in [E]^{\leq \exp k}$). Then $|E| \leq 2^k$.*

In [6], R. Hodel defined a new cardinal invariant for a given Hausdorff space X, namely it was said that $H\psi(x) \leq \tau$ if for every $x \in X$ there is a family $\mathcal{U}_x \in [\exp X]^{\leq \tau}$ of neighborhoods of x such that if $x \neq y$ there are $U_x \in \mathcal{U}_x$ and $U_y \in \mathcal{U}_y$ such that $U_x \cap U_y = \phi$. Obviously, $\psi(X) \leq H\psi(X) \leq \chi(X)$. He also gave an example of a Hausdorff hereditarily Lindelof space X with $H\psi(X) = \omega$ and $t(X) > \omega$. One of his main corollaries from the results about k-structures was the improvement of Archangelskii's inequality to $|X| \leq \exp L(X) \cdot H\psi(X)$.

Corollary 9. *If X is a Hausdorff topological space then:*

$$|X| \leq \exp kaL(X) \cdot H\psi(X).$$

Proof. Let X be a Hausdorff space for which $kaL(X) \cdot H\psi(X) = k$. For each $p \in X$ let $\{V(p,\gamma) : \gamma \in k\}$ be a collection of open neighborhoods of p such that if $p \neq q$ then there exists $\alpha, \beta \in k$ such that $V(p,\alpha) \cap V(q,\beta) = \phi$. We may assume that $\{V(p,\gamma) : \gamma \in k\}$ is closed under finite intersections. Then $\mathcal{V} = \{V(p,\gamma) : \gamma \in k, p \in X\}$ is a k-structure on X satisfying the conditions of Theorem 2.

Let X be a given set and for every $x \in X$ let us have a filter $\Phi_x \subseteq \exp X$ such that $x \in \bigcap \Phi_x$. If $\Phi = \{\Phi_x : x \in X\}$ then the pair (X, Φ) is called a neighborhood space, [14]. With any neighborhood space (X, Φ), a closure operator $[\ ]^\Phi$ could be linked i.e. for $A \subseteq X$ let $[A]^\Phi = \{x \in X : U_x \cap A \neq \phi$

for every $U_x \in \Phi_x\}$. In that sense, every neighborhood space could be looked at as closure space in the sense of [2]. In [10], [11] and [12], cardinal invariants for such spaces are defined and studied.

If (X, Φ) is an o-Hausdorff neighborhood space, we could define $H\psi_o(X)$ as the smallest infinite cardinal τ such that for every $p \in X$ there is a collection $\mathcal{U}_x \subseteq \Phi_x$ with $|\mathcal{U}_x| \leq \tau$ such that if $x \neq y$ there exist $A \in \mathcal{U}_x$ and $B \in \mathcal{U}_y$ such that $A \cap B = \phi$. Let us also note that for an o-Hausdorff neighborhood space (X, Φ) one always has that $\psi_o(X) \leq H\psi_o(X) \leq \chi_o(X)$ and that if $H\psi_{\tau(\Phi)}(X)$ is the corresponding cardinal number for the topological space $(X, \tau(\Phi))$ generated on X by the neighborhood structure Φ then $H\psi_{\tau(\Phi)}(X) \geq H\psi_o(X)$.

If $kL_o(X)$ is the corresponding invariant for neighborhood spaces introduced and studied in [11], we get the following:

Corollary 10 [11]. *For every o-Hausdorff neighborhood space* (X, Φ) *we have that* $|X| \leq \exp kL_o(X) \cdot H\psi_o(X)$.

Proof. Let (X, Φ) be an o-Hausdorff neighborhood space. Let $kL_o(X) \cdot H\psi_o(X) = k$ and for every $p \in X$ let us take the collection $\{V(p, \gamma) : \gamma \in k\}$ of elements of Φ_p, closed under finite intersections in such a way that if $p \neq q$ then there are $\alpha, \beta \in k$ such that $V(p, \alpha) \cap V(q, \beta) = \phi$. Then $\mathcal{V} = \{V(p, \gamma) : \gamma \in k, p \in X\}$ is a k-structure on X satisfying the conditions of Theorem 2. Therefore $|X| \leq 2^k$.

Corollary 11 [11]. *For every Hausdorff topological space* X *we have that* $|X| \leq \exp kL(X) \cdot H\psi(X)$.

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Received 17.02.1993