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New Analytic Solution of the Trinomial Algebraic Equation $z^n + pz + q = 0$ by means of the Goursat Hypergeometric Function, I

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In this paper a representation of the branches of the n-valued algebraic function of the equation $\zeta^n - n\zeta + (n-1)t = 0$ by means of the Goursat hypergeometric function and the Pochhammer integral is given. In particular, for $n = 3$, the branches of the three-valued algebraic function of the cubic equation $\zeta^3 - 3\zeta - 2t = 0$ is investigated in details by means of the Gauss hypergeometric function and the Euler integral.

1. Introduction

In a series of classical works, Capelli, Mellin, G. Belardinelli, and Birkeland develop methods to find the analytic solution of the general algebraic equations representing an arbitrary power of the roots (or the roots themselves when the exponent is equal to unit) by means of definite integrals and infinite power series which by other Birkeland and Belardinelli's methods are reduced to exactly determined finite sums of generalized hypergeometric functions of several variables, composed by the coefficients of the equations. In G. Belardinelli's classical memoir [1], alongside with the basic general results of these authors concerning the problem, more particular results or certain general ones of their predecessors or contemporaries are also given (e. g. for the equation of fifth degree, the trinomial equations, etc.). A bibliography, unique in its contents and completeness is suggested there as well. At the present time F. G. Kravchenko [2,3,4,5] in his own way has again obtained (with other notations) the series for the whole powers of the roots of general algebraic equations, and the representations of these series by means of finite sums of generalized hypergeometric functions (compare these results from [3] to the corresponding results from [1, p. 47: the Mellin series (21) and pp. 54–58: the Birkeland theorem]). An essentially new element in the works of F. G. Kravchenko [3,5] is the investigation of the groups of substitutions of indices in the series for the first powers of the roots as an apparatus for analytic continuation of these series in other domains as well as for establishing a relation between solutions of the general algebraic equations and the trinomial equations. Therefore the trinomial equations have always occupied a central place and have served as a starting point for the investigations of all authors mentioned above.

Birkeland [1, pp. 56–58] redduces the Mellin series for an arbitrary power of the roots of general trinomial algebraic equation of degree n, reduced to Mellin one-parameter form, to a sum of n full Goursat hypergeometric functions of order $n-1$ with one and the same argument proportional to the nth power of the parameter. For the whole power of the roots , this result has been again obtained by F. G. Kravchenko [3, pp. 778–779]. In this work, we find one class of trinomial one-parameter algebraic equations of degree n for which the first powers of the roots are represented analytically and, what is more, univalently (the problem of univalence of the solutions is not considered by the authors mentioned) as a sum of $n-1$ full Goursat hypergeometric functions of order $n-2$ with one and the same argument equal to the $(n-1)$th power of the parameter of the equation. This class is obtained by the trinomial equations

$$z^n + pz + q = 0 \qquad (p \neq 0,\ n \geq 2) \tag{1}$$

by means of our transformations $(\arg(-p) = \theta;\ -\pi < \theta \leq \pi)$

$$\begin{cases} z = \zeta\left(\sqrt[n-1]{-\frac{p}{n}}\right)_0 & \left(\arg\left(\sqrt[n-1]{-\frac{p}{n}}\right)_0 = \frac{\theta}{n-1}\right), \\ t = \frac{q}{(n-1)\left(\left(\sqrt[n-1]{-\frac{p}{n}}\right)_0\right)^n} & \left(\arg\left(\left(\sqrt[n-1]{-\frac{p}{n}}\right)_0\right)^n = \frac{n\theta}{n-1}\right), \end{cases} \tag{2}$$

where we have taken the principal value of the radical for definiteness (the principal value of each radical of order ν $(\nu \geq 2)$ lies in the angle $(-\pi/\nu, \pi/\nu]$ and we shall denote it by the sign 0), which reduce them to the needed new one-parameter form

$$\zeta^n - n\zeta + (n-1)t = 0 \qquad (n \geq 2;\ \forall t). \tag{3}$$

It can be verified by the Birkeland method improved by us (see the proof of the Theorem 1 below) that the class (3) only tolerates lowering the number, the order and the power of the argument, as we noted above, of the full Goursat hypergeometric functions in the analytic represented of first power of roots. On the other hand, the class (1) for $2 \le n \le 5$ appears as a representative of all algebraic equations from the second to the fifth degree since they can always be reduced to this form by means of the classical algebraic transformations: for the third-degree equation by the Horner transformation (the case $p \neq 0$ can always be obtained by an additional linear-fractional transformation), and for the fourth-degree and the fifth-degree equations by the Tschirnhaus and the Jerrard transformation, respectively. Hence, the new general analytic and univalent theory of the reduced to canonical form algebraic equations from second to fifth degree (3), as well as of the equations of an arbitrary degree of the form (3) constructed by us below, can be used as a new foundation in the analytical theory of algebraic functions.

2. Representation of the branches of the n-valued algebraic function of equation $\zeta^n - n\zeta + (n-1)t = 0$ by means of the Goursat hypergeometric function and the Pochhammer integral

It follows from (3) that the n-valued algebraic function $\zeta = f(t)$, supplying the solutions of this equation, is inverse of the polynomial

$$t = \varphi(\zeta) = \frac{n}{n-1}\zeta - \frac{1}{n-1}\zeta^n \qquad (n \ge 2). \tag{4}$$

The zeroes of the derivative $\varphi'(\zeta)$ in the ζ-plane are

$$\epsilon_k = \exp i\,\frac{2k\pi}{n-1} \qquad (k = 0, 1, \dots, n-2;\ \epsilon_k^{n-1} = 1), \tag{5}$$

and their images $\eta_k = \varphi(\epsilon_k)$ in the t-plane determine the finite branch points of the first order of the function $\zeta = f(t)$:

$$\eta_k = \epsilon_k = \exp i\,\frac{2k\pi}{n-1} \qquad (k = 0, 1, \dots, n-2). \tag{6}$$

The point $t = \infty$ is a branch point of order $n-1$. Since under the mapping by the polynomial (4) each ϵ_k is a corresponding double η_k-point $(0 \le k \le n-2)$ then, separating the factor $(\zeta - \epsilon_k)^2$ of the equation (3) for $t = \eta_k$, we obtain that the remaining $n-2$ simple η_k-points are the roots of the equation

$$\sum_{r=0}^{n-2} (r+1)\epsilon_k^r \zeta^{n-2-r} = 0 \qquad (n \ge 3;\ 0 \le k \le n-2). \tag{7}$$

Hence, for each finite $t \neq \eta_k$ $(0 \leq k \leq n-2)$ the roots of equation (3) are distinct. We shall obtain the solution of the equation (7) as well as its remarkable relation with the binomial equation (see (37) and Theorem 2, respectively).

We discover the following basic relation between the roots of equation (3) and the roots of the $(n-1)$-th power of unity. If $\zeta_0 = f_0(t)$ is a solution of (3) (for brevity here and further on we do not write an additional index n) then

$$\zeta_k = f_k(t) = \epsilon_k f_0(\frac{t}{\epsilon_k}) \qquad (0 \leq k \leq n-2) \tag{8}$$

are solutions of (3), too, since

$$f_k^n(t) - n f_k(t) + (n-1)t = \epsilon_k \left[f_0^n(\frac{t}{\epsilon_k}) - n f_0(\frac{t}{\epsilon_k}) + (n-1)(\frac{t}{\epsilon_k}) \right] = 0. \tag{9}$$

We shall call the p r i n c i p a l s o l u t i o n of equation (3) a solution $\zeta_0 = f_0(t)$ for which we assume that for $t = 0$ its value is $\zeta_0^0 = f_0(0) = \sqrt[n-1]{n}_+$ (by the sign + we shall denote arithmetic values of real radicals). Then all solutions (8) for $t = 0$ have the values

$$\zeta_k^0 = f_k(0) = \epsilon_k \sqrt[n-1]{n}_+ \qquad (0 \leq k \leq n-2). \tag{10}$$

The fruitful idea of defining the principal analytic solution of the algebraic equations as well as of proving the existence of a relation between their roots and the roots of unity originates from M e l l i n [1, pp. 37, 42].

If we cut the t-plane along the rays $g_k : [\eta_k, \eta_k \cdot \infty]$ $(0 \leq k \leq n-2)$, going from η_k to ∞ radially with respect to the origin, we shall obtain the simply-connected region $G : \{t \notin g_k : 0 \leq k \leq n-2\}$ (or identically $G : \{t^{n-1} \notin [1, +\infty]\}$), in which, according to the algebraic function theorem, n distinct regular and univalent branches of the n-valued function $\zeta = f(t)$ will be separated. If the branches for $k = 0, 1, \ldots, n-2$ are given by the values (10) then they are represented by the functions (8), and the branch for $k = n-1$ is represented by the functions

$$\begin{cases} \zeta_{n-1} = f_{n-1}(t) = -\sum_{k=0}^{n-2} \epsilon_k f_0(\frac{t}{\epsilon_k}) & (n \geq 3), \\ \zeta_1 = f_1(t) = 2 - f_0(t) & (n = 2), \end{cases} \tag{11}$$

respectively, which for $t = 0$ have values

$$\zeta_{n-1}^0 = f_{n-1}(0) = 0 \qquad (n \geq 2). \tag{12}$$

Thus, thanks to (8) and (11), we obtain the principally important result that the n-valued algebraic function $\zeta = f(t)$ is determined completely only by its single branch $\zeta_0 = f_0(t)$. Analogously, we shall call it the **principal branch**. If we find its element at the point $t = 0$ and realize an analytic continuation of the element into the region G then according to the monodromy theorem we shall obtain the explicit analytic representation of the principal branch $\zeta_0 = f_0(t)$ (i.e. of the principal solution $\zeta_0 = f_0(t)$) in G. Thus the n-th roots of the equation (3) will be represented analytically and univalently by the branches $\zeta_k = f_k(t)$ $(0 \leq k \leq n-1)$ in G and by their analytic continuations on the banks of the cuts g_k $(0 \leq k \leq n-2)$. Therefore, we shall call the region G the **fundamental region** of the equation (3).

Before formulating our basic proposition, we shall discuss one more question, related to the solution on the equation (7). The function (4) realizes an one-to-one mapping of the segment $1 \leq \zeta \leq \sqrt[n-1]{n}_+$ onto the segment $1 \geq t \geq 0$. Hence, $f_0(1) = 1$. This equation, (8), and (6) give us a more general equation $f_k(\eta_k) = \epsilon_k$ $(0 \leq k \leq n-2)$, in consequence to which we shall call the point $\eta_0 = 1$ the **principal branch point** of the algebraic function $\zeta = f(t)$. Hence, the branches which supply the double root $\zeta = \epsilon_k$ of the equation (3) for $t = \eta_k$ are $\zeta_k = f_k(t)$ and $\zeta_{n-1} = f_{n-1}(t)$, i.e.

$$\epsilon_k = f_k(\eta_k) = f_{n-1}(\eta_k) \qquad (0 \leq k \leq n-2), \tag{13}$$

and the remaining simple roots of (3) (for $t = \eta_k$), i.e. the roots of the equation (7) are

$$\zeta_{sk} = f_s(\eta_k) = \epsilon_s f_0(\frac{\eta_k}{\epsilon_s}) \qquad (s \neq k;\ 0 \leq s \leq n-2;\ n \geq 3). \tag{14}$$

Further, for an arbitrary x we make use of the Pochhammer symbol

$$\begin{cases} (x)_m = x(x+1)\ldots(x+m-1) = \frac{\Gamma(x+m)}{\Gamma(x)} & (m = 1, 2, \ldots), \\ (x)_0 = 1; \quad (1)_m = m!\,. \end{cases} \tag{15}$$

Now we are able to prove the following propositions:

Theorem 1. I. *For $n \geq 3$ in the fundamental region G, the regular and univalent branches $\zeta_k = f_k(t)$ $(0 \leq k \leq n-1)$ of the n-valued algebraic function $\zeta = f(t)$ of the trinomial equation (3) have the analytic representation*

$$\begin{cases} \zeta_k = f_k(t) = -\sum_{r=0}^{n-2} a_{rk} t^r F_r(t^{n-1}) & (0 \leq k \leq n-2), \\ \zeta_{n-1} = f_{n-1}(t) = \frac{(n-1)t}{n} F_1(t^{n-1}), \end{cases} \tag{16}$$

where

$$(17) \qquad a_{rk} = \frac{\prod_{s=1}^{r-1}[(n-1)s+r-1]}{n^r\, r!\,(\epsilon_k \sqrt[n-1]{n_+})^{r-1}} \qquad (0 \le r, k \le n-2),$$

where the empty product $\prod_{s=1}^{r-1}$ *for* $r = 0, 1$ *is replaced by* $(-1)^{r-1}$ *and*

$$(18) \quad \begin{cases} F_r(t^{n-1}) = F\begin{pmatrix} \alpha_{r1}, \ldots, \alpha_{r,n-1} & \\ & t^{n-1} \\ \beta_{r1}, \ldots, \beta_{r,n-2} & \end{pmatrix} & (0 \le r \le n-2), \\ \alpha_{rs} = \frac{r-1}{n-1} + \frac{s}{n} & (s = 1, 2, \ldots, n-1;\ 0 \le r \le n-2), \\ \beta_{rs} = \frac{r+s}{n-1} & (s = 1, 2, \ldots, n-r-2;\ 0 \le r \le n-3), \\ \beta_{rs} = \frac{r+s+1}{n-1} & (s = n-r-1, n-r, \ldots, n-2;\ 1 \le r \le n-2) \end{cases}$$

are the principal branches of the full Goursat hypergeometric functions of order $n-2$ *represented for* $|t| \le 1$ *by the absolutely and uniformly convergent identically-named hypergeometric series*

$$(19) \qquad F_r(t^{n-1}) = \sum_{m=0}^{\infty} \frac{\prod_{s=1}^{n-1}(\alpha_{rs})_m}{\prod_{s=1}^{n-2}(\beta_{rs})_m} \cdot \frac{t^{(n-1)m}}{m!} \qquad (0 \le r \le n-2),$$

which are analytic continuation into G by the Pocchammer integrals

$$(20)\ F_r(t^{n-1}) = c_r \int_0^1 \!\!\ldots\! \int_0^1 (1 - t^{n-1} \prod_{s=1}^{n-2} \tau_s)^{-\alpha_{r,n-1}} \prod_{s=1}^{n-2} \tau_s^{\alpha_{rs}-1}(1-\tau_s)^{\beta_{rs}-\alpha_{rs}-1} d\tau_s,$$

$$c_r = \frac{\prod_{s=1}^{n-2}\Gamma(\beta_{rs})}{\prod_{s=1}^{n-2}\Gamma(\alpha_{rs})\Gamma(\beta_{rs}-\alpha_{rs})} \qquad (0 \le r \le n-2)$$

for the principal values of the powers meaningful at the branch points (6) as well, i. e. for $t^{n-1} = 1$.

II. *For* $n = 2$ *we have*

$$(21) \qquad \begin{cases} \zeta_0 = f_0(t) = 2 - \frac{t}{2}F(\frac{1}{2}, 1; 2; t) = 1 + (\sqrt{1-t})_0, \\ \zeta_1 = f_1(t) = \frac{t}{2}F(\frac{1}{2}, 1; 2; t) = 1 - (\sqrt{1-t})_0, \end{cases}$$

respectively, where F is the principal branch of the Gauss hypergeometric function represented for $|t| \le 1$ *by the absolutely and uniformly convergent identically-named hypergeometric series*

$$(22) \qquad F(\frac{1}{2}, 1; 2; t) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m}{(2)_m} t^m,$$

which are analytic continuation for $t \notin [1, +\infty]$ by the Euler integral

$$(23) \qquad F(\tfrac{1}{2}, 1; 2; t) = \int_0^1 \frac{d\tau}{(\sqrt{1-\tau t})_0} = \frac{2}{1+(\sqrt{1-t})_0}\,.$$

Proof. For $n \geq 2$ in the neighborhood of the point $\zeta_0^0 = \sqrt[n-1]{n}_+$ the polynomial (4) has Taylor expansion

$$(24) \quad t = \varphi(\zeta) = -n(\zeta - \sqrt[n-1]{n}_+) - \frac{1}{n-1}\sum_{\nu=2}^{n}\binom{n}{\nu}(\sqrt[n-1]{n}_+)^{n-\nu}(\zeta - \sqrt[n-1]{n}_+)^{\nu}.$$

Inverting (24), according to our general formula for the inversion of the power series [6, p. 233, Theorem 9], we obtain the element of the principal solution $\zeta_0 = f_0(t)$ at the point $t = 0$ by means of the absolutely and uniformly convergent for $|t| \leq 1$ power series

$$(25) \qquad \zeta_0 = f_0(t) = -\sum_{\nu=0}^{\infty} \frac{\prod_{s=1}^{\nu-1}[(n-1)s+\nu-1]}{n^{\nu}\,\nu!\,(\sqrt[n-1]{n}_+)^{\nu-1}}\, t^{\nu} \equiv -\sum_{\nu=0}^{\infty} a_{\nu} t^{\nu}\,,$$

where the empty product $\prod_{s=1}^{\nu-1}$ for $\nu = 0, 1$ is replaced by -1 and 1, respectively.

Birkeland, Belardinelli, Kampé de Fériet [1, pp. 54–58] and F. G. Kravchenko [3, pp. 778–779] sum up the Mellin series for the roots of the algebraic equations of degree n in the Mellin form, replacing the summing index ν linearly by n. Fot the equation (1), reduced in our form (3), the series (25) for $n \geq 3$ is summed up in a simpler way replacing the index ν linearly by $n-1$, i. e. we shall set

$$(26) \qquad \nu = m(n-1) + r \qquad (m = 0, 1, 2, \ldots;\ 0 \leq r \leq n-2;\ n \geq 3).$$

So we obtain

$$(27) \qquad \zeta_0 = f_0(t) = -\sum_{r=0}^{n-2} t^r \sum_{m=0}^{\infty} a_{m(n-1)+r}\, t^{(n-1)m} \qquad (n \geq 3),$$

where

$$(28) \qquad \frac{a_{(m+1)(n-1)+r}}{a_{m(n-1)+r}} = \frac{\prod_{s=1}^{n-1}(m + \frac{r-1}{n-1} + \frac{s}{n})}{\prod_{s=1}^{n-1}(m + \frac{r+s}{n-1})} \quad (m \geq 0;\ 0 \leq r \leq n-2;\ n \geq 3).$$

The correctness of (28) for $m = 0$ and $r = 0, 1$ follows, if we replace the empty products in a_r by $(-1)^{r-1}$, respectively, while for the product in a_{n-1+r} we have in mind the identity

$$(29)\quad \prod_{s=2}^{n+r-1} [(n-1)s+r-1] = (-1)^{r-1}n \prod_{s=1}^{n-1}[(n-1)s+(r-1)n] \quad (r=0,1;\ n\geq 3),$$

verified directly for $r = 0, 1$ after we replace s for $s + r - 1$ on the left-hand side.

Hence, we have

$$(30)\quad a_{m(n-1)+r} = a_r \frac{\prod_{s=1}^{n-1}(\frac{r-1}{n-1} + \frac{s}{n})_m}{\prod_{s=1}^{n-1}(\frac{r+s}{n-1})_m} \quad (m \geq 0;\ 0 \leq r \leq n-2;\ n \geq 3),$$

where

$$(31)\quad a_r = \frac{\prod_{s=1}^{r-1}[(n-1)s+r-1]}{n^r\, r!\,(\sqrt[n-1]{n_+})^{r-1}} \quad (0 \leq r \leq n-2;\ n \geq 3)$$

and the empty product $\prod_{s=1}^{r-1}$ for $r = 0, 1$ is replaced by $(-1)^{r-1}$, respectively.

Thus, the principal analytic and univalent solution (27) of the trinomial equation (3) ($n \geq 3$) acquires the closed form

$$(32)\quad \zeta_0 = f_0(t) = -\sum_{r=0}^{n-2} a_r t^r F_r(t^{n-1}) \quad (n \geq 3),$$

where the analytic continuation of the series (19) into the region G is represented by the real Pochhammer integrals (20) (obviously, this analytic continuation can be also realized by the complex Mellin integrals) [1, pp. 9–11]. The above-mentioned convergence of the series (19) and the integrals (20) follows from the relation

$$(33)\quad \sum_{s=1}^{n-2}\beta_{rs} - \sum_{s=1}^{n-1}\alpha_{rs} = (r+\frac{n-2}{2}) - (r+\frac{n-3}{2}) = \frac{1}{2} \quad (0\leq r\leq n-2; n\leq 3).$$

Now, from (32), (8) and (11) the representations (16) of all branches $\zeta_k = f_k(t)$ (($0 \leq k \leq n-1$; $n \geq 3$) follow, in which the general coefficients (17) take up the place of (31). In particular, $a_{r0} = a_r$ $(0 \leq r \leq n-2;\ n \geq 3)$.

For $n = 2$ the series (25) is summed by the same method. Replacing the index ν in it with m $(m \geq 0)$, the formula (28) is valid also for $n = 2$ $(r = 0)$ if $m \geq 1$. Hence,

$$(34)\quad a_m = a_1 \frac{(\frac{1}{2})_{m-1}}{(2)_{m-1}} \quad (m \geq 1;\ a_1 = \frac{1}{2}).$$

Thus, the principal analytic and univalent solution (25) of the quadratic equation (3) ($n = 2$) takes the first closed form (21), where the series (22) is continued analytically into the fundamental region $t \notin [1, +\infty]$ by the integral (23). From here and from the second formula in (11) the representation of the second branch in (21) follows as well.

This completes the proof of Theorem 1.

In the general theory of Goursat hypergeometric function, formulas representing in a simple closed form its value (when it exists) for an argument, equal to 1, are not known. Now, we shall give linear relations between such values of these functions, when the parameters are rational numbers of the form (18). Indeed, setting $t = \eta_k$ $(0 \leq k \leq n-2)$ in (16), by means of (13) and (6) we obtain

$$\sum_{r=0}^{n-2} a_r F_r(1) = -1 \qquad (n \geq 3), \tag{35}$$

$$F_1(1) = F\left(\begin{matrix} \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n} \\ \frac{2}{n-1}, \frac{3}{n-1}, \ldots, \frac{n-2}{n-1}, \frac{n}{n-1} \end{matrix} \; 1\right) = \frac{n}{n-1} \qquad (n \geq 3). \tag{36}$$

The values $F_r(1)$ are positive and $0 < F_0(1) < 1$, $F_r(1) > 1$ $(1 \leq r \leq n-2;\ n \geq 3)$.

From (14), (16)–(17) and (31) we find the roots (all of them simple) of the equation (7) for each fixed k $(0 \leq k \leq n-2;\ n \geq 3)$:

$$\zeta_{sk} = -\epsilon_s \sum_{r=0}^{n-2} \left(\frac{\epsilon_k}{\epsilon_s}\right)^r a_r\, F_r(1) \qquad (s \neq k;\ 0 \leq s \leq n-2;\ n \geq 3). \tag{37}$$

If we estimate (37) by means of (35) then we find out that these roots lie in the disc $|\zeta| > 1$.

Introducing the magnitude $x_{sk} = \epsilon_k / \zeta_{sk}$ $(s \neq k;\ 0 \leq s \leq n-2;\ n \geq 3)$, from (7) and (37) we obtain the following new result for another remarkable equation:

Theorem 2. *Let us have for $n \geq 3$ the equation*

$$\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \ldots + x + 1 = 0. \tag{38}$$

Then the roots of the derivative equation

$$(n-1)x^{n-2} + (n-2)x^{n-3} + \ldots + 2x + 1 = 0 \tag{39}$$

are all simple, lying in the disc $|x| < 1$ and represented as rational-fractional functions of order $n-2$ of the ratio of any arbitrary root of the $(n-1)$-th power of unity to all remaining ones by the formulas

$$(40) \qquad x_{sk} = -\frac{\frac{\epsilon_k}{\epsilon_s}}{\sum_{r=0}^{n-2}\left(\frac{\epsilon_k}{\epsilon_s}\right)^r a_r F_r(1)} \qquad (s \neq k;\ 0 \leq s \leq n-2)$$

for all fixed k $(0 \leq k \leq n-2)$, where $\epsilon_k, \epsilon_s, a_r$ and $F_r(1)$ are determined by (5), (31) and (18)–(20) for $t^{n-1} = 1$, respectively.

In particular, for $k = 0$ $(\epsilon_0 = 1)$ we have (as we call it basic ordering of the roots)

$$(41) \qquad x_s \equiv x_{s0} = -\frac{\frac{1}{\epsilon_s}}{\sum_{r=0}^{n-2}\left(\frac{1}{\epsilon_s}\right)^r a_r F_r(1)} \qquad (s = 1, \ldots, n-2).$$

The relationship between an arbitrary ordering (40) and the basic ordering (41) is

$$(42) \qquad \begin{cases} x_{s+k,k} = x_{s,0} & (1 \leq s \leq n-k-2;\ 0 \leq k \leq n-3), \\ x_{s+k-n+1,k} = x_{s,0} & (n-k-1 \leq s \leq n-2;\ 1 \leq k \leq n-2). \end{cases}$$

Hence, the roots of the equation (39) can be tabulated for such values of n, which are encountered in practice, by any of the solutions (40) (for example, by the basic solution (41)). For that purpose the constants $F_r(1)$ $(0 \leq r \leq n-2)$ ($F_1(1)$ is given from (36)) should be calculated with the necessary exactness by the series (19) (or by the integrals (20)) for $t^{n-1} = 1$. The equality (35) can be used for verification of the exactness.

From Theorem 1, it follows that the analytic solution of the equation (3), besides for $n = 2$, is expressed also for $n = 3$ by the Gauss hypergeometric function (see Theorem 3 below), while for $n = 4$ it is expressed by the Clausen hypergeometric function, to which the Goursat hypergeometric functions (18) are reduced for $n = 3$ and $n = 4$, respectively. For $n \geq 5$, the analytic solution of the equation (3) is expressed in an integrally linear way by means of the superior Goursat hypergeometric functions.

Hence, to Theorem 1 for $n = 3$, we can apply the detailed theory of the Gauss hypergeometric function (see this theory in [1] and in greater detail in [7] or in other handbooks). In this way we discover a sequence of new important representations as well as properties of the roots of the cubic equation, by which we may consider that its analytic theory is constructed completely.

3. Application: Investigation of the analytic solution of the cubic equation $\zeta^3 - 3\zeta - 2t = 0$

3.1. Representation of the branches of the three-valued algebraic function in the fundamental region by means of the Gauss hypergeometric function and the Euler integral. For $n = 3$ we replace t by $-t$. Thus from (1)–(4) we obtain

$$z^3 + pz + q = 0 \quad (p \neq 0), \tag{43}$$

$$\left\{ \begin{aligned} z &= \zeta\left(\sqrt{-\tfrac{p}{3}}\right)_0 \quad \left(\arg\left(\sqrt{-\tfrac{p}{3}}\right)_0 = \tfrac{\theta}{2};\ \arg(-p) = \theta;\ -\pi < \theta \leq \pi\right), \\ t &= -\frac{q}{2\left(\left(\sqrt{-\frac{p}{3}}\right)_0\right)^3} \quad \left(\arg\left(\left(\sqrt{-\tfrac{p}{3}}\right)_0\right)^3 = \tfrac{3\theta}{2}\right), \end{aligned} \right. \tag{44}$$

$$\zeta^3 - 3\zeta - 2t = 0 \quad (\forall t), \tag{45}$$

$$t = \varphi(\zeta) = \frac{1}{2}\zeta^3 - \frac{3}{2}\zeta, \tag{46}$$

respectively.

For our further purposes it is necessary to reduce in this item the following results from section 2 for $n = 3$ and the replacement of t by $-t$. The three-valued algebraic function $\zeta = f(t)$ of the equation (45) is inverse to the polynomial (46). According to (6) its branch points in the t-plane are $\eta_{0,1} = \mp 1$ (of the first order) and $t = \infty$ (of the second order). Hence, the cuts in the t-plane are $g_0 : [-\infty, -1]$ and $g_1 : [1, +\infty]$, and the whole cut is $g = g_0 \cup g_1$. The fundamental region of the equation (45) is $G : \{t \notin g\}$ (or identically $G : \{t^2 \notin g_1\}$), in which the three branches $\zeta_k = f_k(t)$ $(k = 0, 1, 2)$ of the function $\zeta = f(t)$ are regular and univalent. They are determined, according to (10) and (12), by the values $\zeta^0_{0,1} = f_{0,1}(0) = \pm\sqrt{3}$ and $\zeta^0_2 = f_2(0) = 0$. The branch point $\eta_0 = -1$ and the branch $\zeta_0 = f_0(t)$ are the principal branch point and the principal branch of the function $\zeta = f(t)$, respectively (the branch $\zeta_0 = f_0(t)$ is the principal solution of the equation (45)). In the mapping by the polynomial (46), each of the points $\eta_{0,1} = \mp 1$ of the t-plane has in the ζ-plane a double original $\epsilon_{0,1} = \pm 1$, respectively, according to (5)–(6), and a simple original $\zeta_{10} = -2$ and $\zeta_{01} = 2$, respectively, according to (7) and the notations in (13)–(14), i.e. $f_{0,2}(-1) = 1$, $f_1(-1) = -2$ and $f_{1,2}(1) = -1$, $f_0(1) = 2$. Hence, for finite $t \neq \eta_{0,1}$ the three roots of the equation (45) are different. The important relations (8) and (11) here concern the symmetric values of the argument with respect to the origin and have the form

$$f_0(t) = -f_1(-t), \qquad f_2(t) = -f_2(-t), \qquad (t \in G), \tag{47}$$

which are also true (as we shall see in the following item) on the banks of the cut g.

For $n = 3$ and the replacement of t by $-t$ Theorem 1 is formulated in the following way:

Theorem 3. *In the fundamental region G, the regular and univalent branches $\zeta_k = f_k(t)$ $(k = 0, 1, 2)$ of the three-valued algebraic function $\zeta = f(t)$ of the cubic equation (45) have the analytic representation*

$$(48)\qquad \begin{cases} \zeta_{0,1} = f_{0,1}(t) = \pm\sqrt{3}\, u_0(t) + u_1(t), \\ \zeta_2 = f_2(t) = -2u_1(t), \end{cases}$$

where

$$(49)\qquad \begin{cases} u_0(t) = F(-\frac{1}{6}, \frac{1}{6}; \frac{1}{2}; t^2), \\ u_1(t) = \frac{t}{3} F(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}; t^2) \end{cases} \qquad (t \in G \cup \{\pm 1\})$$

are the principal branches of the Gauss hypergeometric functions of the first and the second kind, respectively, which for $|t| \leq 1$ are represented by means of the absolutely and uniformly convergent identically-named hypergeometric series

$$(50)\qquad \begin{cases} F(-\frac{1}{6}, \frac{1}{6}; \frac{1}{2}; t^2) = \sum_{m=0}^{\infty} \frac{(-\frac{1}{6})_m (\frac{1}{6})_m}{(\frac{1}{2})_m\, m!}\, t^{2m}, \\ F(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}; t^2) = \sum_{m=0}^{\infty} \frac{(\frac{1}{3})_m (\frac{2}{3})_m}{(\frac{3}{2})_m\, m!}\, t^{2m}, \end{cases}$$

analyticaly continued into G by the Euler integrals

$$(51)\qquad \begin{cases} F(-\frac{1}{6}, \frac{1}{6}; \frac{1}{2}; t^2) = \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})} \int_0^1 \tau^{-5/6}(1-\tau)^{-2/3}(1-\tau t^2)^{1/6}\, d\tau, \\ F(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}; t^2) = \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})} \int_0^1 \tau^{-1/3}(1-\tau)^{-1/6}(1-\tau t^2)^{-1/3}\, d\tau, \end{cases}$$

for the principal values of the powers meaningful at the points $t = \pm 1$ as well.

Now, with the help of Theorem 3, we shall construct the whole analytic theory of the cubic equation (45).

3.2. The Riemann surface of the three-valued algebraic function and continuous continuation of the branches on the banks of cuts. The function (46) maps one-to-one the segments $\zeta \in h_0 : [-\infty, -2]$ and $\zeta \in h_1 : [2, +\infty]$ onto the segments g_0 and g_1, respectively, and maps also the segment $\zeta \in [-2, 2]$ onto the three-times-describable segment $t \in [-1, 1]$. Hence, the segment $h = h_0 \cup h_1$ is one real original of the segment g by the mapping (46). In order to find the other two originals, let us take a certain point $u \in g$, the real original of which is the point $v \in h$, i.e. the equality $v^3 - 3v - 2t = 0$ is notified. Separating the factor $\zeta - v$ of the equation (45) for $t = u$, we obtain the other two originals

$$\zeta = \xi \pm i\eta = -\frac{v}{2} \pm i\,\frac{\sqrt{3(v^2 - u)}}{2} \tag{52}$$

of the point u as functions of the real original v. From here, excepting v, we find the hyperbola

$$H : \xi^2 - \frac{\eta^2}{3} = 1 \tag{53}$$

in the ζ-plane with vertices ± 1 and foci ± 2, whose two symmetric parts of the right branch H_0 and of the left branch H_1, respectively, are the other two originals of the segments g_0 and g_1, respectively.

Let us denote by D_0 and D_1 the interiors of the branches H_0 and H_1, respectively, and by D_2 the exterior of the hyperbola H. In the extended ζ-plane, the D_k $(k = 0, 1, 2)$ are three maximal domains of univalence of the polynomial (46), and hence, they represent the domains of the values of the three regular and univalent branches $\zeta_k = f_k(t)$ $(k = 0, 1, 2)$ of the inverse function $\zeta = f(t)$ in their domains of existence in the t-plane. Now, we shall determine these domains. Circuiting the banks of the cuts along g of the three sheets of the Riemann surface of the function $\zeta = f(t)$ superimposed on the fundamental region G, we are convinced that they are "glued" in the following way: Along cut g_0, the lower and the upper bank of the third sheet $(k = 2)$ are "glued on" the upper and the lower bank of the first sheet $(k = 0)$, respectively, and the two banks of the second sheet $(k = 1)$ are "glued together", i.e. on the second sheet the cut along g_0 vanishes. Along cut g_1 the lower and the upper bank of the third sheet are "glued on" the upper and the lower bank of the second sheet, respectively, and the two banks of the first sheet are "glued together", i.e. on the first sheet the cut along g_1 vanishes. Making this curcuit, we build simultaneously the image of the Riemann surface in the extended ζ-plane as well, composed of the three domains D_k $(k = 0, 1, 2)$, after "gluing" the two banks of the cut along the segment h and the corresponding banks of the cut along the hyperbola H. Hence, the domains of existence of the branches $\zeta_k = f_k(t)$ $(k = 0, 1, 2)$ are the domains $t \notin g_0$, $t \notin g_1$ and $t \notin g$ (i.e. G), respectively, which they map regularly and univalently onto the domains D_k $(k = 0, 1, 2)$, respectively. From here it follows that when considering the branches $\zeta_0 = f_0(t)$ and $\zeta_1 = f_1(t)$ only in the fundamental region G, in the domains D_0 and D_1, cuts along the segments h_1 and h_0, respectively, should be made. Then, at symmetric points of the banks of the cut g with respect to the origin the branches $\zeta_k = f_k(t)$ $(k = 0,1,2)$ fulfil the relations (47).

More particularly, from the results obtained, it follows that the equation (45) has real roots for $t \in [-1,1]$ only, and these roots lie on the segment $\zeta \in [-2,2]$. On the other hand, for $t \in (-1,1)$ the Cardano formula for the three real roots of the equation (45) is, generally, irreducible to real radicals and it is known by the Hölder classic investigations. Hence, we can consider the "irreducible case" in a new aspect, and more generally for $t \in [-1,1]$, as a unique case when the roots of the equation (45) are real. For the equation (43) the condition $p<0$ sholud be joined as well (see Theorem 4 in Part II, by which we uncover the geometric meaning of the case $t \in [-1,1]$ for an arbitrary complex $p \neq 0$).

Further, from the construction of the Riemann surface and the formulas (52) and (47), important relations between the boundary values of the branches $\zeta_k = f_k(t)$ $(k = 0,1,2)$ at opposite points of the banks of the cut g follow. Namely, on the banks of the cut g_1 we have the relations

$$(54)\quad \begin{cases} \zeta_0^+ = f_0(t+i0) = f_0(t-i0) = -2\Re\zeta_2^+, & \\ \zeta_1^+ = f_1(t+i0) = f_2(t-i0) = \bar{\zeta}_2^+, & (t \in g_1) \\ \zeta_2^+ = f_2(t+i0) = f_1(t-i0) = \bar{\zeta}_1^+, & (\Re\zeta_2^+ < 0;\ \Im\zeta_2^+ \leq 0,\ t=1), \end{cases}$$

and at the symmetric points with respect to the origin lying on the banks of the cut g_0, the relations

$$(55)\quad \begin{cases} \zeta_0^- = f_0(-t-i0) = f_2(-t+i0) = -\bar{\zeta}_2^+, \\ \zeta_1^- = f_1(-t-i0) = f_1(-t+i0) = -2\Re\zeta_2^+, \quad (-t \in g_0) \\ \zeta_2^- = f_2(-t-i0) = f_0(-t+i0) = -\zeta_2^+. \end{cases}$$

The magnitudes $\zeta_k^\pm$ $(k = 0,1,2)$ express the corresponding boundary values of the branches $\zeta_k = f_k(t)$ $(k = 0,1,2)$, continuously continued on the banks of the cut g, and hence, represent the roots of the equation (45) for $t \in g$. In order to find these magnitudes, it is sufficient to find the magnitude $\zeta_2^+ = f_2(t+i0)$.

Theorem 4. *At symmetric points of the banks of the cut g with respect to the origin, the boundary values* $\zeta_k^{\pm}$ $(k = 0, 1, 2)$ *of the three-valued algebraic function* $\zeta = f(t)$ *of the cubic equation (45) have the analytic representation*

$$(56)\qquad \begin{cases} \zeta_0^+ = -\zeta_1^- = 2u_0((\sqrt{1-t^2})_0), & (t \in g_1) \\ \zeta_{1,2}^+ = -\zeta_{0,2}^- = -u_0((\sqrt{1-t^2})_0) \pm \sqrt{3}\, u_1((\sqrt{1-t^2})_0), & \end{cases}$$

where $u_{0,1}((\sqrt{1-t^2})_0)$ *are obtained from (49)–(51) after the replacement of t by* $(\sqrt{1-t^2})_0 = i\sqrt{t^2-1}_+$.

Proof. Let us transform the second formula in (48) by means of the basic functional relations of the Gauss hypergeometric function [7]. In the domains $\{\Re t <> 0;\ t \notin g\}$ (to which we can add the boundary points $t = \pm 1$ as well), the formula

$$(57)\ F(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}; t^2) = \frac{3}{2} F(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; 1-t^2) - \frac{\sqrt{3}}{2}((\sqrt{1-t^2})_0) F(\frac{7}{6}, \frac{5}{6}; \frac{3}{2}; 1-t^2)$$

is valid, which is obtained if we apply to the function on its left-hand side the transformation additional to unity of the argument. Besides, by a transformation of the parameters on the right-hand side, we have

$$(58)\qquad \begin{cases} F(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; 1-t^2) = \pm \frac{1}{t} F(-\frac{1}{6}, \frac{1}{6}; \frac{1}{2}; 1-t^2), & \\ F(\frac{7}{6}, \frac{5}{6}; \frac{3}{2}; 1-t^2) = \pm \frac{1}{t} F(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}; 1-t^2) & (\Re t <> 0), \end{cases}$$

where the signs $+$ and $-$ correspond to $\Re t > 0$ and $\Re t < 0$. By means of (57)–(58), the second formula in (48) attains the form

$$(59)\ \ \zeta_2 = f_2(t) = \mp u_0((\sqrt{1-t^2})_0) \pm \sqrt{3}\, u_1((\sqrt{1-t^2})_0)\ \ (\{\Re t <> 0;\ t \notin g\}),$$

where $u_{0,1}((\sqrt{1-t^2})_0)$ are obtained from (49)–(51) after the replacement of t by the radical $(\sqrt{1-t^2})_0$ and the upper signs and the lower signs correspond to $\Re t > 0$ and $\Re t < 0$. Hence, the continuous continuation of the branch $\zeta_2 = f_2(t)$ on the banks of the cut g is realized by the continuous continuation of the radical $(\sqrt{1-t^2})_0$, entering into the function $u_1((\sqrt{1-t^2})_0)$. For that purpose, according to (54)–(55) it is sufficient to continue continuously this radical on any bank of the cut g_1. Circuiting the point $t = 1$ in the upper (lower) half-plane, the radical obtains the factor $-i$ (i) on the upper (lower) bank of g_1. Hence, after the circuit we shall have

$$(60)\qquad u_1((\sqrt{1-(t \pm i0)^2})_0) = \mp u_1((\sqrt{1-t^2})_0) \qquad (t \in g_1).$$

From here and from (59) we obtain the boundary value

$$(61)\quad \zeta_2^+ = f_2(t+i0) = -u_0((\sqrt{1-t^2})_0) - \sqrt{3}\,u_1((\sqrt{1-t^2})_0) \quad (t \in g_1).$$

By means of (61) from (54)–(55) we obtain all formulas (56).
This completes the proof of Theorem 4.

By means of Theorem 4 from (54)–(55) we obtain the jumps of the branches by the transition through the cut g:

$$(62)\qquad \begin{aligned} f_{1,2}(t+i0) - f_{1,2}(t-i0) &= f_{0,2}(-t+i0) - f_{0,2}(-t-i0) \\ &= \pm 2\sqrt{3}u_1((\sqrt{1-t^2})_0)\ (t \in g_1). \end{aligned}$$

For obtaining the formulas (56) we made use of the formulas (54)–(55), which follow by the geometric consideration in the beginning of this item. In principle, it is important to note that the formulas (54)–(55) follow also in an analytic way from the formulas (48), if we apply the transformation additional to unity of the argument and of the two hypergeometric functions in (49). Then, continuously continuing the radical $(\sqrt{1-t^2})_0$ on all banks of the cut g, we obtain all boundary values $f_k(t \pm i0)$ $(k = 0,1,2;\ t \in g)$. Their comparison give again the formulas (54)–(55), from which, conversely, we can construct immediately the Riemann surface of the function $\zeta = f(t)$ as well. Further, we shall obtain for a number of times other analytic deductions of the formulas (54)–(55).

3.3. Representation of the three branches in their regions of existence by means of the Gauss hypergeometric functions and the Euler integral. The following proposition gives an analytic determination of the branches in their regions of existence, in which they are regular and univalent:

Theorem 5. *In their own regions of existence, the regular and univalent branches $\zeta_k = f_k(t)$ $(k = 0,1,2)$ of the three-valued algebraic function $\zeta = f(t)$ of the cubic equation (45) have the analytic representation*

$$(63)\qquad \begin{cases} \zeta_0 = f_0(t) = 2u_2(t) & (t \notin g_0), \\ \zeta_1 = f_1(t) = -2u_2(-t) & (t \notin g_1), \\ \zeta_2 = f_2(t) = 2[u_2(-t) - u_2(t)] & (t \in G), \end{cases}$$

where

$$(64)\qquad u_2(\pm t) = F(-\frac{1}{3}, \frac{1}{3}; \frac{1}{2}; \frac{1 \mp t}{2}) \qquad (\pm t \notin g_0 \setminus \{-1\})$$

are the principal branches of the Gauss hypergeometric functions of first kind represented for $|t \mp 1| \leq 2$ *by means of the absolutely and uniformly convergent identically-named hypergeometric series*

$$F(-\frac{1}{3},\frac{1}{3};\frac{1}{2};\frac{1\mp t}{2}) = \sum_{m=0}^{\infty} \frac{(-\frac{1}{3})_m\,(\frac{1}{3})_m}{(\frac{1}{2})_m\; m!}\left(\frac{1\mp t}{2}\right)^m, \tag{65}$$

which are analytically continued into $\pm t \notin g_0$ *by the Euler integrals*

$$F(-\frac{1}{3},\frac{1}{3};\frac{1}{2};\frac{1\mp t}{2}) = \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})}\int_0^1 \tau^{-2/3}(1-\tau)^{-5/6}(1-\tau\frac{1\mp t}{2})^{1/3}d\tau, \tag{66}$$

respectively, for the principal values of the powers meaningful at the points $t = \mp 1$ *as well.*

The branch $\zeta_2 = f_2(t)$ *is represented by the difference of the two series (65) in the moon* $\{|t-1| \leq 2\} \cap \{|t+1| \leq 2\}$, *analytically continued into* G *by the difference of the two integrals (66).*

Proof. By the formula for the quadratic transformation of the Gauss hypergeometric function [7] for $|t| \leq 1$ we have the equality

$$u_2(-t) = \frac{\sqrt{3}}{2}u_0(t) - \frac{1}{2}u_1(t), \tag{67}$$

from which we obtain another equality by replacing t with $-t$. Subtracting these two equalities, we obtain a third equality. By these three equalities the formulas (48) are transformed into the formulas (63). Continuing them analytically into the corresponding cut t-plane, we obtain the assertions of Theorem 5.

Appllying the transformation additional to unity of any of the arguments $(1\mp t)/2$ of the hypergeometric function (64), we can reduce the right-hand side of (63) to a dependence on one of these arguments only; for example, on $(1-t)/2$. This transformation represents again the branches in the fundamental region G and the originating boundary values on the banks of the cut g by means of the Gauss hypergeometric function, in which the argument depends already integrally linearly on t.

Theorem 6. *In the fundamental region* G *and in symmetric points of the banks of cut* g *with respect to the origin, the regular and univalent branches* $\zeta_k = f_k(t)$ $(k = 0,1,2)$ *and the boundary values* $\zeta_k^{\pm}$ $(k = 0,1,2)$ *of the three-valued algebraic function* $\zeta = f(t)$ *of the cubic equation (45) have the corresponding analytic representations*

$$\begin{cases} \zeta_0 = f_0(t) = 2u_2(t), \\ \zeta_{1,2} = f_{1,2}(t) = -u_2(t) \mp \sqrt{3}\,u_3(t), \end{cases} \quad (t \in G) \tag{68}$$

$$(69)\qquad \begin{cases} \zeta_0^+ = -\zeta_1^- = 2u_2(t), \\ \zeta_{1,2}^+ = -\zeta_{0,2}^- = -u_2(t) \pm \sqrt{3}\,u_3(t), \end{cases} \qquad (t \in g_1)$$

where $u_2(t)$ is the Gauss hypergeometric function of first kind (64)–(66) and

$$(70)\qquad u_3(t) = \frac{2}{3}\Big(\sqrt{\frac{1-t}{2}}\Big)_0 F\Big(\frac{1}{6},\frac{5}{6};\frac{3}{2};\frac{1-t}{2}\Big) \qquad (t \notin g_0\backslash\{-1\})$$

is its corresponding Gauss hypergeometric function of second kind represented for $|t-1| \le 2$ by the absolutely and uniformly convergent identically-named hypergeometric series

$$(71)\qquad F\Big(\frac{1}{6},\frac{5}{6};\frac{3}{2};\frac{1-t}{2}\Big) = \sum_{m=0}^{\infty} \frac{(\frac{1}{6})_m\,(\frac{5}{6})_m}{(\frac{3}{2})_m\, m!}\Big(\frac{1-t}{2}\Big)^m,$$

which is analytically continued for $t \notin g_0$ by the Euler integral

$$(72)\; F\Big(\frac{1}{6},\frac{5}{6};\frac{3}{2};\frac{1-t}{2}\Big) = \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{6})\,\Gamma(\frac{2}{3})}\int_0^1 \tau^{-1/6}(1-\tau)^{-1/3}\Big(1-\tau\frac{1-t}{2}\Big)^{-1/6} d\tau,$$

for the principal values of the powers meaningful at the point $t = -1$ as well.

Proof. The branch point $t = 1$ of the function $\zeta = f(t)$ is a branch point of the quadratic radical in (70) as well, so the function (70) will become regular if in the domain $t \notin g_0$ the cut g_1 is made. Besides, in the fundamental region G only (to which we can also add the points $t = \pm 1$) the formula

$$(73)\qquad u_2(-t) = \frac{1}{2}\,u_2(t) + \frac{\sqrt{3}}{2}\,u_3(t)$$

is valid, which is obtained after the transformation additional to unity of the argument $(1+t)/2$ of the hypergeometric function in the left-hand side [7]. With the help of (73), all representations (63) are replaced by the representation (68), from which the second and the third ones are valid in $G \cup \{\pm 1\}$ only. Hence, the continuous continuation of the branches $\zeta_k = f_k(t)$ $(k = 0, 1, 2)$ on the banks of the cut g_1 is realized by the continuous continuation of the quadratic radical in (70). Circuiting the point $t = 1$ in the upper (lower) half-plane, the radical obtains the factor $-i$ (i) on the upper (lower) bank of g_1. Hence, after circuiting we shall have

$$(74)\qquad u_3(t \pm i0) = \mp u_3(t) \qquad (t \in g_1).$$

From here, from (68) and from (54), we obtain the formulas (69) for ζ_k^+ $(k = 0, 1, 2)$ whence and from (55), the formulas (69) for ζ_k^- $(k = 0, 1, 2)$ follow.

This completes the proof of Theorem 6.

With the help of Theorem 6, from (54)–(55), we obtain in another form the jumps of the branches by the transition through the cut g:

$$(75)\qquad \begin{aligned} & f_{1,2}(t+i0) - f_{1,2}(t-i0) \\ &= f_{0,2}(-t+i0) - f_{0,2}(-t-i0) = \pm 2\sqrt{3}u_3(t) \quad (t \in g_1). \end{aligned}$$

The comparison of (75) and (62) gives the equality $u_3(t) = u_1((\sqrt{1-t^2})_0)$ $(t \in g_1)$.

From (68)–(69), the representations depending on the other argument $(1+t)/2$ as well of the hypergeometric function (64)–(66) follow. That's why it is suffficient in (68)–(69) to replace t with $-t$ and to apply the relations (47) to the left-hand side of (68) as well, which, in consequence to the alternation of the signs implies the corresponding substitution of the indices, while in the left-hand side of (69) without changing the indices, to replace the roles of ζ_k^+ and ζ_k^- $(k = 0,1,2)$, since now $t \in g_0$. The replacement of t by $-t$ is applicable also to the formula (75), which becomes valid for $t \in g_0$ without changing the indices.

3.4. Representation of the branches in the cut neighborhood of the point at infinity by means of the Gauss hypergeometric function and the Euler integral. This is realized with the help of the following proposition:

Theorem 7. *In the upper and the lower t-half-plane, the regular and univalent branches $\zeta_k = f_k(t)$ $(k = 0,1,2)$ of the three-valued algebraic function $\zeta = f(t)$ of the cubic equation (45) have analytic representation*

$$(76)\ \zeta_k = f_k(t) = (\sqrt[3]{2t})_k\, u_4(t) + \frac{1}{(\sqrt[3]{2t})_k} u_5(t) \quad (k = 0,1,2;\ 0 < \arg t < \pi),$$

$$(77)\ \zeta_k = f_k(t) = \epsilon^k (\sqrt[3]{2t})_k u_4(t) + \frac{1}{\epsilon^k (\sqrt[3]{2t})_k} u_5(t) \ (k = 0,1,2;\ -\pi < \arg t < 0)$$

respectively, where

$$(78)\quad (\sqrt[3]{2t})_k = \epsilon^k \sqrt[3]{2\,|t|}_+ \exp i\,\frac{\arg t}{3} \quad (k = 0,1,2;\ \epsilon = \exp i\,\frac{2\pi}{3}) \ \textit{and}$$

$$(79)\quad u_4(t) = F(-\frac{1}{6}, \frac{1}{3}; \frac{2}{3}; \frac{1}{t^2}), \quad u_5(t) = F(\frac{1}{6}, \frac{2}{3}; \frac{4}{3}; \frac{1}{t^2}), \quad (t \notin (-1,1))$$

are the principal branches of the Gauss hypergeometric function of first kind represented for $|t| \geq 1$ *by the absolutely and uniformly convergent identically-named hypergeometric series*

$$(80)\qquad \begin{cases} F(-\frac{1}{6},\frac{1}{3};\frac{2}{3};\frac{1}{t^2}) = \sum_{m=0}^{\infty} \frac{(-\frac{1}{6})_m(\frac{1}{3})_m}{(\frac{2}{3})_m\, m!}\,\frac{1}{t^{2m}}, \\ F(\frac{1}{6},\frac{2}{3};\frac{4}{3};\frac{1}{t^2}) = \sum_{m=0}^{\infty} \frac{(\frac{1}{6})_m(\frac{2}{3})_m}{(\frac{4}{3})_m\, m!}\,\frac{1}{t^{2m}}, \end{cases}$$

which are analyticaly continued for $t \notin [-1,1]$ *by the Euler integrals*

$$(81)\qquad \begin{cases} F(-\frac{1}{6},\frac{1}{3};\frac{2}{3};\frac{1}{t^2}) = \frac{\Gamma(\frac{2}{3})}{\Gamma^2(\frac{1}{3})}\int_0^1 \tau^{-2/3}(1-\tau)^{-2/3}(1-\frac{\tau}{t^2})^{1/6}\,d\tau, \\ F(\frac{1}{6},\frac{2}{3};\frac{4}{3};\frac{1}{t^2}) = \frac{\Gamma(\frac{4}{3})}{\Gamma^2(\frac{2}{3})}\int_0^1 \tau^{-1/3}(1-\tau)^{-1/3}(1-\frac{\tau}{t^2})^{-1/6}\,d\tau, \end{cases}$$

for the principal values of the powers meaningful at the points $t = \pm 1$ *as well.*

The values of functions (79) are inverse with respect to each other, i. e.

$$(82)\qquad u_4(t)\,u_5(t) = 1 \qquad (t \notin (-1,1)).$$

Proof. Applying to the hypergeometric function (49) the basic functional relation of the Gauss hypergeometric function for analytic continuation out of the unit disc [7], we shall obtain ($0<|\arg t|<\pi$)

$$(83)\qquad \begin{cases} u_0(t) = 2^{-2/3}(-t^2)^{1/6}u_4(t) + 2^{-4/3}(-t^2)^{-1/6}u_5(t), \\ u_1(t) = 2^{-2/3}t(-t^2)^{-1/3}u_4(t) - 2^{-4/3}t(-t^2)^{-2/3}u_5(t), \end{cases}$$

where the powers are principal. They are calculated by means of the following relations for the principal values of the arguments

$$(84)\qquad \begin{cases} \arg(-t) = \arg t \mp \pi, \\ \arg(-t^2) = 2\arg t \mp \pi, \end{cases}$$

where the signs of the right-hand side are taken $-$ for $0 < \arg t < \pi$ and $+$ for $-\pi < \arg t < 0$. In this way, entering into (48) with (83), we obtain the formulas (76)–(77). For abbreviating the calculation, we shall note that the formulas (77) follow immediately from (76) if we replace t with $-t$, taking into consideration the relation (47). The identity (82) follows from the multiplication of the two series (80) for $|t| \geq 1$ and from the analytic continuation into the domain $t \notin [-1, 1]$ (in order to avoid the calculations see the formulas (41) in section 2, from which the identity (82) follows immediately).

This completes the proof of Theorem 7.

From Theorems 5 and 7 it follows that the two formulas (76)–(77) for $k = 0$ and for $k = 1$ are continued analytically each one in the other through the cuts $(1, +\infty)$ and $(-\infty, -1)$, respectively, and, at that, immediately, since on their banks the right-hand sides of the formulas considered coincide. With this analytic continuation, the formulas (76)–(77) for $k = 0, 1$ will represent the branches $\zeta_{0,1} = f_{0,1}(t)$ in the entire t-plane cut along the segments $t \in [-\infty, 1]$ and $t \in [-1, +\infty]$, respectively. In this way, particularly, the formulas (76)–(77) $(k = 0, 1, 2)$ will represent the expansions of the branches $\zeta_k = f_k(t)$ $(k = 0, 1, 2)$ in the neighborhood $|t| \geq 1$ at the point $t = \infty$ cut along g_0, g_1 and g, respectively.

Finally, from the formulas (76)–(77) corresponding representations of the boundary values $\zeta_k^{\pm}$ $(k = 0, 1, 2)$ as well follow immediately and by this and those of the jumps through cut g.

We set Theorems 3–7 in the foundation of the representations of the branches $\zeta_k = f_k(t)$ $(k = 0, 1, 2)$ and of the boundary values $\zeta_k^{\pm}$ $(k = 0, 1, 2)$ of the three-valued algebraic function $\zeta = f(t)$ of the cubic equation (45) by means of the Gauss hypergeometric function. From them we can obtain a sequence of other representations by means of this function, applying further its basic functional relations [7] in the already explained way. Besides, we can always choose such functional relations which help us to obtain other quick convergent hypergeometric series with the radius of convergence 1 around the points 0 or 1, the analytic continuation of which are obtained by the corresponding Euler integrals.

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