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Two Mappings in Connection to Jensen's Inequality

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Refinements of Jensen's discrete inequality by the use of the mappings introduced in (1) and (4) and certain applications in connection with some well-known inequalities in Mathemetical Analysis are given.

1. Introduction

Let I be an interval of real numbers and $f : I \to \mathbb{R}$ be a convex mapping on I. The following inequality is known in literature as Jensen's inequality:

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i f(x_i),$$

where x_i are in I, p_i are nonnegative real numbers, $i = 1, 2, \dots, n$ and $\sum_{i=1}^{n} p_i = 1$.

In this note, using mappings (1) and (4), we propose some refinements of Jensen's inequality. Certain applications are also given.

For other recent results in connection with this inequality, we refer to the papers [3–7] where further references are given.

2. The main results

For a given convex function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and for $x_i \in I$, $p_i \ge 0$

$(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^n p_i = 1$, we shall consider the mapping

$$H : [0,1] \to \mathbb{R}, \quad H(t) := \sum_{i=1}^n p_i f\left(tx_i + (1-t)\sum_{j=1}^n p_j x_j\right). \tag{1}$$

The following theorem holds.

Theorem 1. *If $f : I \to \mathbb{R}$ is as above, then*
(i) H is convex on $[0,1]$;
(ii) We have the bounds

$$\begin{aligned} \inf_{t\in[0,1]} H(t) &= H(0) = f\left(\sum_{i=1}^n p_i x_i\right) \\ \sup_{t\in[0,1]} H(t) &= H(1) = \sum_{i=1}^n p_i f(x_i); \end{aligned} \tag{2}$$

(iii) H is increasing on $[0,1]$.

Proof. (i). Obvious by the convexity of d.

(ii). We shall prove the following inequalities:

(3)
$$f\left(\sum_{i=1}^n p_i x_i\right) \le H(t) \le t\sum_{i=1}^n p_i f(x_i) + (1-t) f\left(\sum_{i=1}^n p_i x_i\right) \le \sum_{i=1}^n p_i f(x_i)$$

for all t in $[0,1]$.

By Jensen's inequality, we have:

$$H(t) \ge f\left(\sum_{i=1}^n p_i\left[tx_i + (1-t)\sum_{j=1}^n p_j x_j\right]\right) = f\left(\sum_{i=1}^n p_i x_i\right).$$

Now, using the convexity of f, we get:

$$H(t) \le \sum_{i=1}^n p_i[tf(x_i) + (1-t)f(\sum_{j=1}^n p_j x_j)] = t\sum_{i=1}^n p_i f(x_i) + (1-t)f(\sum_{i=1}^n p_i x_i),$$

and the second inequality in (3) is also proved.

The last inequality is obvious observing that the mapping

$$g(t) := t\sum_{i=1}^n p_i f(x_i) + (1-t) f(\sum_{i=1}^n p_i x_i)$$

is increasing on $[0,1]$.

(iii). Let t_1, t_2 be in $(0,1)$ with $t_2 > t_1$. Since H is convex on $(0,1)$, we have:

$$\frac{H(t_2)-H(t_1)}{t_2-t_1} \geq H'_+(t_1) = \sum_{i=1}^{n} p_i f'_+(t_1 x_i + (1-t_1)\sum_{j=1}^{n} p_j x_j)(x_i - \sum_{j=1}^{n} p_j x_j)$$

where $f'_+(x_0)$ denotes the right-hand derivative of f in the point x_0.

Since f is convex on I, we have:

$$f(x) - f(y) \geq f'_+(y)(x-y)$$

for all x, y in $\mathring{I}$ ($\mathring{I}$ is the interior of I), which implies that:

$$f\left(\sum_{i=1}^{n} p_i x_i\right) - f\left(t_1 x_i + (1-t_1)\sum_{j=1}^{n} p_j x_j\right) \geq$$

$$\geq t_1 f'_+\left(t_1 x_i + (1-t_1)\sum_{j=1}^{n} p_j x_j\right)\left(\sum_{j=1}^{n} p_j x_j - x_i\right).$$

Consequently, we have

$$f'_+\left(t_1 x_i + (1-t_1)\sum_{j=1}^{n} p_j x_j\right)\left(x_i - \sum_{j=1}^{n} p_j x_j\right) \geq$$

$$\frac{1}{t_1}\left[f\left(t_1 x_i + (1-t_1)\sum_{j=1}^{n} p_j x_j\right) - f\left(\sum_{i=1}^{n} p_i x_i\right)\right].$$

By multiplying with $p_i \geq 0$ and summing over $i = 1, 2, \ldots, n$, we deduce that:

$$H'_+(t_1) \geq \frac{1}{t_1}(H(t_1) - H(0)) \geq 0 \qquad \text{if } t_1 \in (0,1),$$

which shows that H is increasing on $(0,1)$ and, by (ii), also in $[0,1]$.

Thus the proof is finished.

Now, we shall introduce the second mapping in connection to Jensen's inequality.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping on the interval I. Consider the mapping

$$(4) \qquad F : [0,1] \to \mathbb{R}, \quad F(t) := \sum_{i,j=1}^{n} p_i p_j f(t x_i + (1-t) x_j)$$

where $p_i \geq 0$ with $\sum_{i=1}^{n} p_i = 1$ and $x_i \in I$ $(i = 1, 2, \ldots, n)$.

The following theorem holds.

Theorem2. *Let f, p_i and x_i be as above. Then*
(i) $F(s+1/2) = F(1/2-s)$ *for all s in* $[0,1/2]$;
(ii) F *is convex on* $[0,1]$;
(iii) We have the bounds:

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \sum_{i=1}^n p_i f(x_i);$$

$$\inf_{t\in[0,1]} F(t) = F(1/2) = \sum_{i,j=1}^n p_i p_j f(\tfrac{x_i+x_j}{2});$$

(iv) The following inequality is valid:

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq F(1/2);$$

(v) F *is decreasing on* $[0,1/2]$ *and increasing on* $[1/2,1]$;
(vi) The inequality:

$$H(t) \leq F(t) \qquad \text{for all } t \text{ in } [0,1] \text{ holds.}$$

Proof. (i). Obvious by definition of F.
(ii). Follows by the convexity of f.
(iii). Since f is convex on I, hence:

$$f(tx_i + (1-t)x_j) \leq tf(x_i) + (1-t)f(x_j)$$

for all $i,j = 1,2,\ldots,n$ and t in $[0,1]$. By multiplying with $p_i p_j \geq 0$ and summing over $i,j = 1,2,\ldots,n$, we get:

$$\sum_{i,j=1}^n f(tx_i + (1-t)x_j)p_i p_j \leq \sum_{i,j=1}^n [tf(x_i) + (1-t)f(x_j)]p_i p_j = \sum_{i=1}^n p_i f(x_i),$$

which shows that $F(t) \leq F(0) = F(1)$, for all t in $[0,1]$.

On the other hand, by the convexity of f, we also have:

$$\frac{1}{2}\left[f(tx_i + (1-t)x_j) + f(tx_j + (1-t)x_i)\right] \geq f\left(\frac{x_i+x_j}{2}\right)$$

for all $i,j = 1,2,\ldots,n$ and t in $[0,1]$. By multiplying this inequality with $p_i p_j \geq 0$ and summing over $i,j = 1,2,\ldots,n$, we derive:

$$\sum_{i,j=1}^n p_i p_j f(\frac{x_i+x_j}{2}) \leq \frac{1}{2}\sum_{i,j=1}^n p_i p_j \left[f(tx_i + (1-t)x_j) + f((1-t)x_i + tx_j)\right]$$

$$= \sum_{i,j=1}^n p_i p_j\, f(tx_i + (1-t)x_j),$$

which implies that $f(1/2) \le F(t)$ for all t in $[0,1]$, and the statement is proved.

(iv). Using Jensen's inequality, we have:

$$\sum_{i,j=1}^{n} p_i p_j f(\frac{x_i + x_j}{2}) \ge f\left(\sum_{i,j=1}^{n} p_i p_j (\frac{x_i + x_j}{2})\right) = f\left(\sum_{i=1}^{n} p_i x_i\right),$$

which proves the statement.

(v). Since F is convex on $(0,1)$, we have for $t_2 > t_1$, $t_2, t_1 \in (1/2, 1)$:

$$\frac{F(t_2) - F(t_1)}{t_2 - t_1} \ge F'_+(t_1) = \sum_{i,j=1}^{n} f'_+(t_1 x_i + (1-t)x_j)(x_i - x_j) p_i p_j .$$

By the convexity of f on I, we deduce:

$$f(\frac{x_i + x_j}{2}) - f(t_1 x_i + (1-t_1)x_j) \ge \frac{1}{2} f'_+(t_1 x_i + (1-t_1)x_j)(x_i - x_j)(1 - 2t_1)$$

for all $i, j = 1, 2, \ldots, n$ and $t_1 \in (1/2, 1)$.

By multiplying with $p_i p_j \ge 0$ and summing over $i, j = 1, 2, \ldots, n$, we derive that:

$$F'_+(t_1) \ge \frac{2}{2t_1 - 1}(F(t_1) - F(1/2)) \ge 0, \quad t_1 \in (1/2, 1),$$

which shows that F is increasing on $(1/2, 1)$ and, by (iii), also in $[1/2, 1]$.

The fact that F is decreasing on $[0, 1/2]$ goes likewise and we omit it.

(vi). A simple computation shows that:

$$H(t) = \sum_{i=1}^{n} p_i f\left(\sum_{j=1}^{n} [t x_i + (1-t)x_j] p_j\right) .$$

Using Jensen's inequality, we derive that

$$H(t) \le \sum_{i,j=1}^{n} p_i p_j f(t x_i + (1-t)x_j) = F(t), \quad t \in [0,1]$$

and the proof of the theorem is finished.

Now, we will give some applications of these main results.

3. Applications

1. Let $x_i, p_i \geq 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^n p_i = 1$. Then the following refinements of arithmetic mean – geometric mean hold:

$$\sum_{i=1}^n p_i x_i \geq \prod_{i=1}^n \left(t x_i + (1-t) \sum_{j=1}^n p_j x_j \right)^{p_i} \geq \prod_{i=1}^n x_i^{p_i}$$

and

$$\sum_{i=1}^n p_i x_i \geq \prod_{i,j=1}^n \left(\frac{x_i + x_j}{2} \right)^{p_i p_j} \geq \prod_{i,j=1}^n (t x_i + (1-t) x_j)^{p_i p_j} \geq \prod_{i=1}^n x_i^{p_i}$$

for all t in $[0,1]$.

The proof is obvious from Theorem 1 and 2 for the convex mapping $f(x) = -\ln x$, $x > 0$.

2. Let $x_i \in \mathbb{R}$, $p_i \geq 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^n p_i = 1$ and $p \geq 1$. Then the inequalities:

$$\left| \sum_{i=1}^n p_i x_i \right|^p \leq \sum_{i=1}^n p_i \left| t x_i + (1-t) \sum_{j=1}^n p_j x_j \right|^p \leq \sum_{i=1}^n p_i |x_i|^p$$

and

$$\left| \sum_{i=1}^n p_i x_i \right|^p \leq \sum_{i,j=1}^n p_i p_j \left| \frac{x_i + x_j}{2} \right|^p \leq \sum_{i,j=1}^n p_i p_j |t x_i + (1-t) x_j|^p \leq \sum_{i=1}^n p_i |x_i|^p$$

are valid for all t in $[0,1]$.

The proof is obvious from Theorem 1 and 2 for the convex mapping $f(x) = |x|^p$, $p \geq 1$.

3. Let $x_i \in (0, 1/2]$ $(i = 1, 2, \ldots, n)$. Then the following refinements of the well known inequality due to Ky Fan [2, p. 5] are valid:

$$\begin{aligned} \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1 - x_i)} &\geq \prod_{i=1}^n \left[\frac{t x_i + (1-t)\, 1/n \sum_{j=1}^n x_j}{1 - t x_i - (1-t)\, 1/n \sum_{j=1}^n x_j} \right]^{1/n} \\ &\geq \left[\frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n (1 - x_i)} \right]^{1/n} \end{aligned}$$

and

$$\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1-x_i)} \geq \left[\prod_{i,j=1}^n \frac{x_i+x_j}{2-x_i-x_j}\right]^{1/n^2}$$

$$\geq \left[\prod_{i,j=1}^n \left(\frac{tx_i+(1-t)x_j}{1-tx_i-(1-t)x_j}\right)\right]^{1/n^2} \geq \left[\frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n (1-x_i)}\right]^{1/n}$$

for all t in $[0,1]$.

The proof follows by Theorem 1 and 2 for the convex mapping $f(x) = -\ln(\frac{x}{1-x})$, $x \in (0,1/2]$.

Remark. If we choose in Theorem 1 and 2 $f(x) = \ln(\frac{x}{1-x})^x$, $x \in (0,1)$, we obtain a refinement of an intertesting fact established by H. Alzer in [1] which gives a converse inequality for Ky Fan's main result.

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