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Non-Commutative Neutrix Convolution Products of Functions

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The non-commutative neutrix convolution product of the functions $x^r e_-^{\lambda x}$ and $x^s e_+^{\mu x}$ is evaluated for $r, s = 0, 1, 2, \dots$ and all λ, μ. Further non-commutative neutrix convolution products are deduced.

In the following we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The convolution product $f * g$ of two distributions f and g in $\mathcal{D}'$ is then usually defined by the equation

$$\langle (f * g)(x), \phi \rangle = \langle f(y), \langle g(x), \phi(x + y) \rangle \rangle$$

for arbitrary ϕ in $\mathcal{D}$, provided f and g satisfy either of the conditions

(a) either f or g has bounded support,

(b) the supports of f and g are bounded on the same side,see Gel'fand and Shilov [5].

Note that if f and g are locally summable functions satisfying either of the above conditions then

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x - t)\, dt = \int_{-\infty}^{\infty} f(x - t)g(t)\, dt. \tag{1}$$

It follows that if the convolution product $f * g$ exists by this definition then

$$f * g = g * f, \tag{2}$$

$$(f * g)' = f * g' = f' * g. \tag{3}$$

This definition of the convolution product is rather restrictive and so the neutrix convolution product was introduced in [2]. In order to define the neutrix convolution product we first of all let τ be a function in $\mathcal{D}$ satisfying the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \leq \tau(x) \leq 1$,
(iii) $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$,
(iv) $\tau(x) = 0$ for $|x| \geq 1$.The function τ_ν is now defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

for $n = 1, 2, \ldots$.

Definition 1. Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the neutrix convolution product $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided that the limit h exists in the sense that

$$\underset{n\to\infty}{\mathrm{N-lim}} \langle f_n * g, \phi \rangle = \langle h, \phi \rangle,$$

for all ϕ in $\mathcal{D}$, where N is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \ \ln^r n \quad (\lambda > 0, \ r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Note that in this definition the convolution product $f_n * g$ is defined in Gel'fand and Shilov's sense, the distribution f_n having bounded support.

The following theorem was proved in [2], showing that the neutrix convolution product is a generalization of the convolution product.

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the neutrix convolution product $f \circledast g$ exists and*

$$f \circledast g = f * g.$$

The next theorem was also proved in [2].

Theorem 2. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution product $f \circledast g$ exists. Then the neutrix convolution product $f \circledast g'$ exists and*

$$(f \circledast g)' = f \circledast g'.$$

Note however that equation (1) does not necessarily hold for the neutrix convolution product and that $(f \circledast g)'$ is not necessarily equal to $f' \circledast g$.

A number of neutrix convolution products have been evaluated. For example, $x_-^\lambda \circledast x_+^r$ see [2], $x_-^{-r} \circledast x_+^s$ see [3], $\ln x_- \circledast \ln x_+$ see [6] and $\ln x_- \circledast x_+^\mu$, $x_-^\mu \circledast \ln x_+$ see [4].

In order to define further neutrix convolution products, we increase our set of negligible functions given in Definition 1 to also include finite linear sums of the functions

$$n^\lambda e^{\mu n} \quad (\mu > 0).$$

We now define the locally summable functions $e_+^{\lambda x}$ and $e_-^{\lambda x}$ by

$$e_+^{\lambda x} = \begin{cases} e^{\lambda x}, & x > 0, \\ 0, & x < 0, \end{cases} \qquad e_-^{\lambda x} = \begin{cases} 0, & x > 0, \\ e^{\lambda x}, & x < 0. \end{cases}$$

It follows that

$$e_-^{\lambda x} + e_+^{\lambda x} = e^{\lambda x}, \quad x^r e_+^{\lambda x} = x_+^r e_+^{\lambda x}, \quad x^r e_-^{\lambda x} = (-1)^r x_-^r e_-^{\lambda x},$$

for $r = 0, 1, 2, \ldots$.

We now prove

Theorem 3. *The neutrix convolution product $(x^r e_-^{\lambda x}) \circledast (x^s e_+^{\mu x})$ exists and*

$$e_-^{\lambda x} \circledast e_+^{\mu x} = \frac{e_+^{\mu x} + e_-^{\lambda x}}{\lambda - \mu}, \tag{4}$$

$$\begin{aligned}(x^r e_-^{\lambda x}) \circledast (x^s e_+^{\mu x}) &= D_\lambda^r D_\mu^s \frac{e_+^{\mu x} + e_-^{\lambda x}}{\lambda - \mu} \\ &= \sum_{i=0}^{s} \binom{s}{i} \frac{(r+s-i)! x^i e_+^{\mu x}}{(\lambda - \mu)^{r+s-i+1}} + \\ &+ \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i} (r+s-i)! x^i e_-^{\lambda x}}{(\lambda - \mu)^{r+s-i+1}},\end{aligned} \tag{5}$$

where $D_\lambda = \partial/\partial\lambda$ *and* $D_\mu = \partial/\partial\mu$, *for* $\lambda \neq \mu$ *and* $r, s = 0, 1, 2, \ldots$, *these neutrix convolution products existing as convolution products if* $\lambda > \mu$ *and*

$$(x^r e_-^{\lambda x}) \circledast (x^s e_+^{\lambda x}) = B(r+1, s+1) x^{r+s+1} e_-^{\lambda x}, \tag{6}$$

where B denotes the Beta function, for all λ *and* $r, s = 0, 1, 2, \ldots$.

Proof. We put $(e_-^{\lambda x})_n = e_-^{\lambda x} \tau_n(x)$, for $n = 1, 2, \ldots$ and suppose first of all that $\lambda \neq \mu$. Since $e_+^{\mu x}$ and $(e_-^{\lambda x})_n$ are locally summable functions with $(e_-^{\lambda x})_n$ having compact support, the convolution product $(e_-^{\lambda x})_n * e_+^{\mu x}$ is defined by equation (1) and so

$$(e_-^{\lambda x})_n * e_+^{\mu x} = \int_{-\infty}^{\infty} (e_-^{\lambda t})_n e_+^{\mu(x-t)} \, dt. \tag{7}$$

When $-n < x < 0$,

$$\int_{-\infty}^{\infty} (e_-^{\lambda t})_n e_+^{\mu(x-t)} \, dt = e^{\mu x} \int_{-n}^{x} e^{(\lambda-\mu)t} \, dt + e^{\mu x} \int_{-n-n^{-n}}^{-n} e^{(\lambda-\mu)t} \tau_n(t) \, dt$$

$$= \frac{e^{\lambda x} - e^{\mu x - (\lambda-\mu)n}}{\lambda - \mu} + O(n^{-n} e^{-(\lambda-\mu)n}). \tag{8}$$

When $x > 0$,

$$\int_{-\infty}^{\infty} (e_-^{\lambda t})_n e_+^{\mu(x-t)} \, dt = e^{\mu x} \int_{-n}^{0} e^{(\lambda-\mu)t} \, dt + e^{\mu x} \int_{-n-n^{-n}}^{-n} e^{(\lambda-\mu)t} \tau_n(t) \, dt$$

$$= \frac{e^{\mu x} - e^{\mu x - (\lambda-\mu)n}}{\lambda - \mu} + O(n^{-n} e^{-(\lambda-\mu)n}). \tag{9}$$

It now follows from equations (7), (8) and (9) that for arbitrary ϕ in $\mathcal{D}$

$$\langle (e_-^{\lambda x})_n * e_+^{\mu x}, \phi(x) \rangle = (\lambda - \mu)^{-1} \langle e_+^{\mu x} + e_-^{\lambda x}, \phi(x) \rangle +$$
$$- (\lambda - \mu)^{-1} e^{-(\lambda-\mu)n} \langle e_+^{\mu x} + e_-^{\mu x}, \phi(x) \rangle + O(n^{-n} e^{-(\lambda-\mu)n})$$

and so

$$\operatorname*{N-lim}_{n\to\infty} \langle (e_-^{\lambda x})_n * e_+^{\mu x}, \phi(x) \rangle = (\lambda - \mu)^{-1} \langle e_+^{\mu x} + e_-^{\lambda x}, \phi(x) \rangle,$$

the usual limit existing if $\lambda > \mu$. Equation (4) follows.

We now put $(x^r e_-^{\lambda x})_n = x^r e_-^{\lambda x} \tau_n(x)$. Then using equation (1) again we have

$$(x^r e_-^{\lambda x})_n * (x^s e_+^{\mu x}) = \int_{-\infty}^{\infty} (t^r e_-^{\lambda t})_n (x-t)^s e_+^{\mu(x-t)} \, dt. \tag{10}$$

When $-n < x < 0$,

$$\begin{aligned}\int_{-\infty}^{\infty}(t^r e_-^{\lambda t})_n(x-t)^s e_+^{\mu(x-t)}\,dt &= e^{\mu x}\int_{-n}^{x} t^r(x-t)^s e^{(\lambda-\mu)t}\,dt+\\ &\quad + e^{\mu x}\int_{-n-n^{-n}}^{-n} t^r(x-t)^s e^{(\lambda-\mu)t}\tau_n(t)\,dt\\ &= D_\lambda^r D_\mu^s e^{\mu x}\int_{-n}^{x} e^{(\lambda-\mu)t}\,dt + O(n^{-n+r+s}e^{-(\lambda-\mu)n})\\ &= D_\lambda^r D_\mu^s \frac{e^{\lambda x}}{\lambda-\mu} + e^{\mu x}P(e^{-(\lambda-\mu)n})+\\ &\quad + O(n^{-n+r+s}e^{-(\lambda-\mu)n}), \qquad (11)\end{aligned}$$

on using equation (8), where P denotes a polynomial.

When $x > 0$,

$$\begin{aligned}\int_{-\infty}^{\infty}(t^r e_-^{\lambda t})_n(x-t)^s e_+^{\mu(x-t)}\,dt &= e^{\mu x}\int_{-n}^{0} t^r(x-t)^s e^{(\lambda-\mu)t}\,dt+\\ &\quad + e^{\mu x}\int_{-n-n^{-n}}^{-n} t^r(x-t)^s e^{(\lambda-\mu)t}\tau_n(t)\,dt\\ &= D_\lambda^r D_\mu^s e^{\mu x}\int_{-n}^{0} e^{(\lambda-\mu)t}\,dt + O(n^{-n+r+s}e^{-(\lambda-\mu)n})\\ &= D_\lambda^r D_\mu^s \frac{e^{\mu x}}{\lambda-\mu} + e^{\mu x}P(e^{-(\lambda-\mu)n})+\\ &\quad + O(n^{-n+r+s}e^{-(\lambda-\mu)n}), \qquad (12)\end{aligned}$$

on using equation (9).

It now follows as above from equations (10), (11) and (12) that for arbitrary ϕ in $\mathcal{D}$

$$\underset{n\to\infty}{\text{N}-\lim}\langle (x^r e_-^{\lambda x})_n * (x^s e_+^{\mu x}), \phi(x)\rangle = D_\lambda^r D_\mu^s(\lambda-\mu)^{-1}\langle e_+^{\mu x} + e_-^{\lambda x}, \phi(x)\rangle,$$

the usual limit existing if $\lambda > \mu$. Thus

$$(x^r e_-^{\lambda x})\circledast(x^s e_+^{\mu x}) = D_\lambda^r D_\mu^s \frac{e_+^{\mu x} + e_-^{\lambda x}}{\lambda-\mu}$$

and equation (5) follows.

Now suppose that $\lambda = \mu$. Then using equation (1) again we have

$$(x^r e_-^{\lambda x})_n * (x^s e_+^{\lambda x}) = \int_{-\infty}^{\infty}(t^r e_-^{\lambda t})_n(x-t)^s e_+^{\lambda(x-t)}\,dt. \qquad (13)$$

When $-n < x < 0$,

$$\int_{-\infty}^{\infty} (t^r e_-^{\lambda t})_n (x-t)^s e_+^{\lambda(x-t)}\, dt =$$

$$= e^{\lambda x} \int_{-n}^{x} t^r (x-t)^s\, dt + e^{\lambda x} \int_{-n-n^{-n}}^{-n} t^r (x-t)^s \tau_n(t)\, dt$$

$$= e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} (-1)^i \int_{-n}^{x} x^{s-i} t^{r+i}\, dt + O(n^{-n+r+s})$$

$$= e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} (-1)^i \frac{x^{r+s+1} - (-n)^{r+i+1} x^{s-i}}{r+i+1} + O(n^{-n+r+s})$$

$$= e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} x^{r+s+1} (-1)^i \int_0^1 t^{r+i}\, dt + e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^r x^{s-i} n^{r+i+1}}{r+i+1} + \\ + O(n^{-n+r+s})$$

$$= B(r+1, s+1) x^{r+s+1} e^{\lambda x} + e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^r x^{s-i} n^{r+i+1}}{r+i+1} + \\ + O(n^{-n+r+s}), \tag{14}$$

where B denotes the Beta function.

When $x > 0$,

$$\int_{-\infty}^{\infty} (t^r e_-^{\lambda t})_n (x-t)^s e_+^{\lambda(x-t)}\, dt =$$

$$= e^{\lambda x} \int_{-n}^{0} t^r (x-t)^s\, dt + e^{\lambda x} \int_{-n-n^{-n}}^{-n} t^r (x-t)^s \tau_n(t)\, dt$$

$$= e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^r x^{s-i} n^{r+i+1}}{r+i+1} + O(n^{-n+r+s}). \tag{15}$$

It now follows as above from equations (13), (14) and (15) that for arbitrary ϕ in $\mathcal{D}$

$$\operatorname*{N-lim}_{n\to\infty} \langle (x^r e_-^{\lambda x})_n * (x^s e_+^{\lambda x}), \phi(x) \rangle = B(r+1, s+1) \langle x^{r+s+1} e_-^{\lambda x}, \phi(x) \rangle$$

and equation (6) follows.

Corollary 1. *The neutrix convolution product* $(x^r e_+^{\lambda x}) \circledast (x^s e_-^{\mu x})$ *exists and*

$$(x^r e_+^{\lambda x}) \circledast (x^s e_-^{\mu x}) = D_\lambda^r D_\mu^s \frac{e_+^{\lambda x} + e_-^{\mu x}}{\mu - \lambda}, \tag{16}$$

for $\lambda \neq \mu$ and $r, s = 0, 1, 2, \dots$, this neutrix convolution product existing as a convolution product if $\lambda > \mu$ and

$$(x^r e_+^{\lambda x}) \circledast (x^s e_-^{\lambda x}) = -B(r+1, s+1) x^{r+s+1} e_+^{\lambda x}, \tag{17}$$

for all λ and $r, s = 0, 1, 2, \dots$.

Proof. Equations (16) and (17) follow immediately on replacing x by $-x$, λ by $-\lambda$ and μ by $-\mu$ in equations (5) and (6) respectively on noting that

$$e_-^{\lambda(-x)} = e_+^{(-\lambda)x}, \quad e_+^{\mu(-x)} = e_-^{(-\mu)x}.$$

Corollary 2. *The neutrix convolution products $(x^r e^{\lambda x}) \circledast (x^s e_\pm^{\mu x})$, $(x^r e_\pm^{\lambda x}) \circledast (x^s e^{\mu x})$ and $(x^r e^{\lambda x}) \circledast (x^s e^{\mu x})$ exist and*

$$(x^r e^{\lambda x}) \circledast (x^s e_\pm^{\mu x}) = \pm D_\lambda^r D_\mu^s \frac{e^{\lambda x}}{\lambda - \mu}, \tag{18}$$

$$(x^r e_\pm^{\lambda x}) \circledast (x^s e^{\mu x}) = \pm D_\lambda^r D_\mu^s \frac{e^{\mu x}}{\lambda - \mu}, \tag{19}$$

$$(x^r e^{\lambda x}) \circledast (x^s e^{\mu x}) = 0, \tag{20}$$

for $\lambda \neq \mu$ and $r, s = 0, 1, 2, \dots$, these neutrix convolution products existing as convolution products if $\lambda > \mu$ and

$$(x^r e^{\lambda x}) \circledast (x^s e_\pm^{\lambda x}) = \pm B(r+1, s+1) x^{r+s+1} e^{\lambda x}, \tag{21}$$

$$(x^r e_\pm^{\lambda x}) \circledast (x^s e^{\lambda x}) = 0, \tag{22}$$

$$(x^r e^{\lambda x}) \circledast (x^s e^{\lambda x}) = 0, \tag{23}$$

for all λ and $r, s = 0, 1, 2, \dots$.

Proof. We will suppose first of all that $\lambda \neq \mu$. Since $x^r e_+^{\lambda x}$ and $x^s e_+^{\mu x}$ are locally summable functions bounded on the same side, the convolution product $(x^r e_+^{\lambda x}) * (x^s e_+^{\mu x})$ is defined by equation (1) and so when $x > 0$, (it is 0 when $x < 0$)

$$(x^r e_+^{\lambda x}) * (x^s e_+^{\mu x}) = \int_0^x t^r e^{\lambda t} (x-t)^s e^{\mu(x-t)}\, dt = D_\lambda^r D_\mu^s \int_0^x e^{\lambda t} e^{\mu(x-t)}\, dt$$

$$= D_\lambda^r D_\mu^s \frac{e_+^{\lambda x} - e_+^{\mu x}}{\lambda - \mu}. \tag{24}$$

Replacing x by $-x$, λ by $-\lambda$ and μ by $-\mu$ in equation (24) we get

$$(x^r e_-^{\lambda x}) * (x^s e_-^{\mu x}) = D_\lambda^r D_\mu^s \frac{e_-^{\lambda x} - e_-^{\mu x}}{\mu - \lambda}. \tag{25}$$

It follows that

$$(x^r e^{\lambda x}) \circledast (x^s e_+^{\mu x}) = [x^r e_+^{\lambda x} + x^r e_-^{\lambda x}] \circledast (x^s e_+^{\mu x}) = D_\lambda^r D_\mu^s \frac{e^{\lambda x}}{\lambda - \mu},$$

on using equations (5) and (24) and noting that the neutrix convolution product is distributive with respect to addition. Similarly,

$$(x^r e^{\lambda x}) \circledast (x^s e_-^{\mu x}) = -D_\lambda^r D_\mu^s \frac{e^{\lambda x}}{\lambda - \mu},$$

on using equations (16) and (25). Equations (18) are proved.

Equations (19) follow similarly from equations (5), (16), (24) and (25) and equation (20) follows immediately from equations (18).

Now suppose that $\lambda = \mu$. Then the convolution product $(x^r e_+^{\lambda x}) * (x^s e_+^{\lambda x})$ is again defined by equation (1) and so

$$\begin{aligned}(x^r e_+^{\lambda x}) * (x^s e_+^{\lambda x}) &= \begin{cases} \displaystyle\int_0^x t^r e^{\lambda t}(x-t)^s e^{\lambda(x-t)}\, dt, & x \geq 0, \\ 0, & x < 0 \end{cases} \\ &= e_+^{\lambda x} \int_0^x t^r (x-t)^s\, dt \\ &= B(r+1, s+1) x^{r+s+1} e_+^{\lambda x}. \end{aligned} \tag{26}$$

Replacing x by $-x$ and λ by $-\lambda$ in equation (26) we get

$$(x^r e_-^{\lambda x}) * (x^s e_-^{\lambda x}) = -B(r+1, s+1) x^{r+s+1} e_-^{\lambda x}. \tag{27}$$

Equations (21), (22) and (23) now follow as above from equations (6), (17), (26) and (27).

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