Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica Balkanica

New series Vol.9, 1995, Fasc.1

On the Spectrum of Disjointness Preserving Operator Semigroups on $C(X, \mathbb{C}^n)$

*Haşmet Gürçay**

Presented by P. Kenderov

It is proved in [4] that the disjointness preservation of an operator $T \in \mathcal{L}(C(X, \mathbb{C}^n))$ is equivalent to existence of a continuous function $Q\colon X \longrightarrow M_n(\mathbb{C})$ and a map $\varphi\colon X \longrightarrow X$ continuous on $X \setminus \{x \in X\colon Q(x) = 0\}$ such that

$$(*) \qquad Tf = Q \cdot f \circ \varphi$$

for every $f \in C(X, \mathbb{C}^n)$.

Using $(*)$ we showed in [3] that if $(T(t))_{t \geq 0}$ is a strongly continuous semigroup on $C(X, \mathbb{C}^n)$ and $1 \otimes e_k \in D(A)$ for each canonical basis vector $e_k \in \mathbb{C}^n$, then the disjointness preservation of $(T(t))_{t \geq 0}$ is equivalent to the existence of a continuous flow $\{\varphi_t\}_{t \geq 0}$ on X and a matrix multiplication operator M on $C(X, \mathbb{C}^n)$, such that

$$T(t)f(x) = \lim_{n \to \infty} \left[\exp \left(\int_0^{\frac{t}{n}} M(\varphi_s(x)) ds \right) \right]^n . f(\varphi_t(x))$$

for all $f \in C(X, \mathbb{C}^n), x \in X$ and $t \geq 0$.

Recall that $C(X, \mathbb{C}^n)$ is the Banach space of all continuous functions on some compact space X endowed with the norm $\|f\| = \sup_{x \in x} |||f(x)|||$ for $f \in C(X, \mathbb{C}^n)$ and some lattice norm $||| \cdot |||$. We denote by $[] \cdot []$ the matrix norm

* The support of DAAD is aknowledged.

generated by $|||\cdot|||$. It is easy to prove that for a disjointness preserving operator $T \in \mathcal{L}(C(X, \mathbb{C}^n))$, which is in the form $(*)$, we have

$$\|T\| = \sup_{x \in X} |||Q(x)|||.$$

In this paper the spectrum of disjointness preserving, strongly continuous semigroups on $C(X, \mathbb{C}^n)$ is investigated. Let us start by giving some well-known definitions and results from spectral theory.

Let E be an arbitrary complex Banach space. We let $\sigma(A)$ denote the spectrum of an operator A on E and $\rho(A) := \mathbb{C} \setminus \sigma(A)$ the resolvent set of A. If $(T(t))_{t \geq 0}$ is a strongly continuous semigroup on E, then there exist constants $C \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Ce^{t\omega}$ for all $t \geq 0$.

Let ω_0 be the *growth bound* of $(T(t))_{t \geq 0}$, that is the infimum of all ω as above. Then

$$\omega_0 = \lim_{t \to \infty} \frac{1}{t} \log \|T(t)\| = \inf_{t>0} \log \|T(t)\|.$$

Let $r(T(t))$ denote the *spectral radius* of $T(t)$, that is

$$r(T(t)) := \sup\{|\lambda| : \lambda \in \sigma(T(t))\},$$

and we employ $s(A)$ to denote the *spectral bound* of $(T(t))_{t \geq 0}$, that is

$$s(A) := \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\}.$$

It is easily seen that for all strongly continuous semigroups $(T(t))_{t \geq 0}$ we have

$$e^{t s(A)} \leq \|T(t)\| \tag{1}$$

for all $t \geq 0$.

Let A be a closed, densely defined operator on E with domain $D(A)$. Now we are making the following definitions.

(i) $P\sigma(A) := \{\lambda \in \mathbb{C} : (\lambda - A)$ is not injective $\}$ *is the point spectrum,*

(ii) $A\sigma(A) := \{\lambda \in \mathbb{C} :$ there exists a normed sequence $(x_n) \subset D(A)$ with $(\lambda - A)x_n \to 0\}$ *is the approximate point spectrum* and

(iii) $R\sigma(A) := \{\lambda \in \mathbb{C} : (\lambda - A)D(A)$ is not dense $\}$ *is the residual spectrum of* A.

From the above definitions we have $P\sigma(A) \subset A\sigma(A)$ and $\sigma(A) = A\sigma(A) \cup R\sigma(A)$.

Let E_0 be a closed, $(T(t))$-invariant subspace of E. Let $\hat{E}: = E/E_0$ be the quotient space with respect to E_0 and let $q: E \to \hat{E}$ be the quotient mapping. Next we consider the induced operators $\hat{T}$ in $\mathcal{L}(\hat{E})$. This operator family yields a strongly continuous semigroup with generator $(\hat{A}, D(\hat{A}))$ where $D(\hat{A}) = q(D(A))$ and $\hat{A}q(x) = q(Ax)$ for $x \in D(A)$. Now let $T(t)_{|E_0} \in L(E_0)$ be the restricted operator on E_0. $(T(t)_{|E_0})_{t\geq 0}$ is strongly continuous semigroup with generator $A_|$ and with domain $D(A_|)$: $\{x \in D(A) \colon Ax \in E_0\}$. Then we have

$$\sigma(A) \subset \sigma(\hat{A}) \cup \sigma(A_|). \tag{2}$$

Let E' be dual space of E and $(T(t)')_{t\geq 0}$ the adjoint semigroup of $(T(t))_{t\geq 0}$ on E. Consider $E^*: = \{\phi \in E' \colon \|\cdot\| - \lim_{t\to 0} T(t)'\phi = \phi\}$. Then $(T(t)^*: = T(t)'_{|E^*})_{t\geq 0}$ is strongly continuous semigroup on E^* with generator $(A^*, D(A^*))$, where $D(A^*): = \{x \in D(A') \colon A'x' \in E^*\}$. Then $\sigma(A) = \sigma(A') = \sigma(A^*)$ and $R\sigma(A) = P\sigma(A') = P\sigma(A^*)$.

For the details we refer to [5] and [6].

For the following we need a matrix theoretical result. It is known that for a given matrix A with the spectral bound $s(A)$ and for given $\varepsilon > 0$, there exists a M_ε such that

$$[]e^{tA}[] \leq M_\varepsilon e^{t(s(A)+\varepsilon)}$$

for all $t \geq 0$

Now we make this result stronger and claim the following lemma.

Lemma 1. *Let K be a relative by compact subset of $M_n(\mathbb{C})$ and $\varepsilon > 0$ a fixed number. Then there exists $M_\varepsilon \geq 1$ such that for all $A \in K$ and $t \geq 0$*

$$[]e^{tA}[] \leq M_\varepsilon e^{t.(s(A)+\varepsilon)} \tag{3}$$

Proof. In view of a Theorem of Schur every matrix $A \in M_n(\mathbb{C})$ is unitary similar to a matrix in upper triangular form. Thus we can assume that every $A \in K$ is an upper triangular form. Thus we can assume that every $A \in K$ is an upper triangular matrix. Now we show the above inequality (3) by induction on n.

The proof is clear for $n = 1$. Suppose that the inequality is true for $n - 1$ and consider the $n \times n$-matrix

$$A = \begin{pmatrix} a & B \\ 0 & D \end{pmatrix}_{1+(n-1),1+(n-1)}$$

Then

$$e^{tA} = \begin{pmatrix} e^{tA} & R(t) \\ 0 & e^{tD} \end{pmatrix}.$$

where $R(t) = \int_0^t e^{\tau a} B e^{(t-\tau)D} d\tau$.

By the our induction hypothesis D is in a relative by compact subset of $M_{n-1}(\mathbb{C})$. Hence there exists c_ε such that

$$[]e^{tD}[] \leq c_\varepsilon e^{t.(s(D)+\varepsilon)} \text{ for all } t \geq 0,$$

where $s(D)$ is the spectral bound of D. Then

$$\begin{aligned}[]R(t)[] &\leq \int_0^t e^{\tau \operatorname{Re} a}[]B[]c_\varepsilon e^{(t-\tau)(s(D)+\varepsilon)} d\tau \\ &= []B[]c_\varepsilon \frac{e^{t.\operatorname{Re} a} - e^{t.(s(D)+\varepsilon)}}{\operatorname{Re} a - s(D) - \varepsilon}.\end{aligned}$$

If $\operatorname{Re} a - s(D) = \varepsilon$, then

$$[]R(t)[] \leq t.e^{t.(s(D)+\varepsilon)}[]B[]c_\varepsilon.$$

In general we have for $\varepsilon \geq 0$

$$(*) \qquad t \leq \frac{1}{\varepsilon} e^{\varepsilon}.t.$$

Thus we obtain

$$[]R(t)[] \leq []B[]c_\varepsilon \frac{1}{\varepsilon} e^{\varepsilon.t} e^{t.(s(D)+\varepsilon)}.$$

Let $M_\varepsilon : = []B[]c_\varepsilon \frac{1}{\varepsilon}$, then

$$[]R(t)[] \leq M_\varepsilon e^{t.(s(D)+2\varepsilon)}.$$

If $\operatorname{Re} a < S(D) + \varepsilon$, then

$$\frac{e^{t.\operatorname{Re} a} - e^{t.(s(D)+\varepsilon)}}{\operatorname{Re} a - s(D) - \varepsilon} \leq t e^{t.(s(D)+\varepsilon)}.$$

If $s(D) + \varepsilon < \operatorname{Re} a$ we find the same result. Then

$$[]R(t)[] \leq []B[]c_\varepsilon t e^{t.(s(D)+\varepsilon)}.$$

By (*) for $\varepsilon > 0$ we have

$$[]R(t)[] \leq []B[]c_\varepsilon \frac{1}{\varepsilon} e^{\varepsilon t} e^{t(s(D)+\varepsilon)}.$$

Let $M_\varepsilon : = []B[]c_\varepsilon \frac{1}{\varepsilon}$. Finally we obtain

$$[]R(t)[] \leq M_\varepsilon e^{t.(s(D)+\varepsilon)}.$$

Proposition 2. *Let $(T(t))_{t\geq 0}$ be a disjointness preserving, strongly continuous semigroup on $C(X,\mathbb{C}^n)$ with generator $(A, D(A))$. Let $1 \otimes e_k \in D(A)$ for all $k \in \{1,...,n\}$. Let (φ_t) and M be the corresponding continuous flow and the matrix multiplicator respectively. For the growth bound ω_o of $(T(t))_{t\geq 0}$ we have*

$$\omega_0 = \lim_{t\to\infty} \sup_{x\in X} \lim_{n\to\infty} s\left(\frac{n}{t}\int_0^{\frac{t}{n}} M(\varphi_\tau(x))d\tau\right),$$

where

$$s\left(\frac{n}{t}\int_0^{\frac{t}{n}} M(\varphi_\tau(x))d\tau\right)$$

is the spectral bound of the matrix

$$\frac{n}{t}\int_0^{\frac{t}{n}} M(\varphi_\tau(x))d\tau.$$

Proof. The conditions of [3, Th4] hold, hence

$$T(t)f(x) = \lim_{n\to\infty}\left(\exp\left(\int_0^{\frac{t}{n}} M(\varphi_\tau(x))d\tau\right)\right)^n .f(\varphi_t(x))$$

for all $f \in E$, where (φ_t) is a continuous flow and M is a matrix multiplicator. The compactness of X and continuity of $M(\cdot)$ imply, that $M(X)$ is a compact subset of $M_n(\mathbb{C})$. On the other hand

$$\frac{n}{t}\int_0^{\frac{t}{n}} M(\varphi_\tau(x))d\tau \in \overline{co}(M(X))$$

for all $x \in X, t \geq 0$, where $\overline{co}(M(X))$ is the closed, convex hull of $M(X)$. Then

$$||T(t)|| = \sup_{x\in X}[] \lim_{n\to\infty}\left(\exp\left(n\cdot\int_0^{\frac{t}{n}} M(\varphi_\tau(x))d\tau\right)\right)[].$$

Let $\varepsilon > 0$ be fixed Then by Lemma 1 there exists $M_\varepsilon \geq 1$ such that

$$\|T(t)\| = \sup_{x\in X}\left(M_\varepsilon.\lim_{n\to\infty}\exp[t.s\left(\frac{n}{t}\int_0^{\frac{1}{n}} M(\varphi_\tau(x))d\tau\right)+t.\varepsilon]\right)$$

for all $t \geq 0$. Therefore

$$(*)\qquad \frac{1}{t}\log\|T(t)\| \leq \frac{1}{t}\log M_\varepsilon + \sup_{x\in X}\lim_{n\to\infty} s\left(\frac{n}{t}\int_0^{\frac{1}{n}} M(\varphi_\tau(x))d\tau\right)+\varepsilon.$$

Furthermore, by using the fact that

$$\|T(t)\| = \sup_{x\in X}[]\lim_{n\to\infty}\exp\left(t.\frac{n}{t}\int_0^{\frac{t}{n}} M(\varphi_\tau(x))d\tau\right)[]$$

and by (1) we have the following inequality

$$(*)\qquad \sup_{x\in X}\lim_{n\to\infty} s\left(\frac{n}{t}\int_0^{\frac{1}{n}} M(\varphi_\tau(x))d\tau)\right.$$
$$\leq \frac{1}{t}\log\sup_{x\in X}[]\lim_{n\to\infty}\exp\left(n.\int_0^{\frac{t}{n}} M(\varphi_\tau(x))d\tau\right)[].$$

By (*) and (**) we get

$$\begin{aligned}\sup_{x\in X}\lim_{n\to\infty} s\left(\frac{n}{t}\int_0^{\frac{1}{n}} M(\varphi_\tau(x))d\tau\right) &\leq \frac{1}{t}\log\|T(t)\|\\ &\leq \frac{1}{t}\log M_\varepsilon\\ &+\sup_{x\in X}\lim_{n\to\infty} s\left(\frac{n}{t}\int_0^{\frac{1}{n}} M(\varphi_\tau(x))d\tau\right)+\varepsilon.\end{aligned}$$

Since $\lim_{t\to\infty}\frac{1}{t}\log M_\varepsilon = 0$, we get the following equality for the growth bound of $(T(t))_{t\geq 0}$,

$$\lim_{t\to\infty}\sup_{x\in X}\lim_{n\to\infty} s\left(\frac{n}{t}\int_0^{\frac{1}{n}} M(\varphi_\tau(x))d\tau\right) = \lim_{t\to\infty}\frac{1}{t}\log\|T(t)\| = \omega_0.$$

This completes the proof.

Corollary 3. *Let $(T(t))_{t\geq 0}$ be a disjointless preserving, strongly continuous semigroup on $C(X < \mathbb{C}^n)$ with generator $(A, D(A))$. Assume that $1 \otimes e_k \in D(A)$ for each canonical basis vector $e_k \in \mathbb{C}^n$. If the matrices $M(x)$, $x \in X$, commute, then the growth bound ω_0 of $(T(t))_{t\geq 0}$ is given by*

$$\omega_0 = \lim_{t\to\infty} \sup_{x\in X} s\left(\frac{1}{t}\int_0^t M(\varphi_\tau(x))d\tau\right)$$

Let us take our disjointness preserving, strongly continuous semigroup $(T(t))_{t\geq 0}$ on $C(X, \mathbb{C}^n)$ and

$$Y: = \bigcap_{t\geq 0} \varphi_t(X),$$

where φ_t is the continuous flow of $(T(t))_{t\geq 0}$ for all $t \geq 0$. Since $\varphi_s(X) \subset \varphi_t(X)$ for $s \geq t$ we obtain $Y \neq \emptyset$ and Y is $\{\varphi_t\}$-invariant. For this reason $J: = \{f \in C(X, \mathbb{C}^n): f(x) = 0 \quad \forall x \in Y\}$ is invariant under the semigroup $(T(t))_{t\geq 0}$.

Next we define

$$\bar{c}_t(M, \varphi): = \sup_{x\in X} \lim_{n\to\infty} s\left(\frac{n}{t}\int_0^{\frac{1}{n}} M(\varphi_\tau(x))d\tau\right).$$

By Proposition 2 we have

$$\bar{c}(M, \varphi): = \lim_{t\to\infty} \bar{c}_t(M, \varphi) = \inf_{t\geq 0} \bar{c}_t(M, \varphi) = \omega_0.$$

From the above

$$-\bar{c}_t(-M, \varphi) = \inf_{x\in X} \lim_{n\to\infty} s\left(\frac{n}{t}\int_0^{\frac{1}{n}} M(\varphi_\tau(x))d\tau\right) = : \underline{c}_t(M, \varphi).$$

Then the following limit and supremum exists

$$\underline{c}_t(M, \varphi): = \lim_{t\to\infty} \underline{c}_t(M, \varphi) = \sup_{t\geq 0} \underline{c}_t(M, \varphi).$$

Now we can give the following propositions.

Proposition 4. *Let $H: = \{\lambda \in \mathbb{C}: \mathrm{Re}\,\lambda < \underline{c}(M, \varphi)\}$. If $\varphi_s(X) = \varphi_t(X)$ for positive numbers s, t, with $s \neq t$, then $H \subseteq R\sigma(A)$ or $H \cap \sigma(A) = \emptyset$.*

Proof. It is obvious that

$$C(X, \mathbb{C}^n)/J \cong C(Y, \mathbb{C}^n),$$

where $Y:=\bigcap_{t\geq 0}\varphi_t(X)$ and $J:=\{f\in E: f(x)=0,\ \forall x\in Y\}$. We let $(\hat{T}(t))_{t\geq 0}$ denote the semigroup induced from $(T(t))_{t\geq 0}$ on $C(Y,\mathbb{C}^n)$ with the generator $(\hat{A},D(\hat{A}))$. For given $\varepsilon>0$ there exists $\rho_0>0$ such that $\underline{c}_t(M,\varphi)\geq \underline{c}(M,\varphi)-\varepsilon$ for all $t\geq\rho_0$. Let $\lambda=\alpha+i\beta$, $\alpha,\beta\in\mathbb{R}$ and $\alpha<\underline{c}(M,\varphi)-4\varepsilon$ for $t\geq\rho_0$. Then $\underline{c}_t(M,\varphi)>\alpha+3\varepsilon$ for $t\geq\rho_0$. By Lemma 1 there exists $M_\varepsilon\geq$ such that

$$\begin{aligned}[]Q_t(x)[] &\leq M_\varepsilon \lim_{n\to\infty}\exp\left(t.\left(s\left(\frac{n}{t}\int_0^{\frac{t}{n}} M(\varphi_\tau(x))\right)+\varepsilon\right)\right)\\ &\leq M_\varepsilon\exp\left(t.(\overline{c}_t(M,\varphi)+\varepsilon)\right),\end{aligned}$$

where $Q_t(x)=\lim_{n\to\infty}\exp(n\int_0^{\frac{t}{n}}M(\varphi_\tau(x))d\tau)$. Therefore, we have the following inequality:

$$\begin{aligned}[]Q(t)[] &\geq []\lim_{n\to\infty}\exp\left(-n\int_0^{\frac{1}{n}}M(\varphi_\tau(x))d\tau\right)[]^{-1}\\ &\geq \frac{1}{M_\varepsilon}\exp(t\underline{c}_t(M,\varphi))-t.\varepsilon)\geq\frac{1}{M_\varepsilon}\exp(t.(\alpha+2\varepsilon))\end{aligned}$$

for all $t>\rho_0$ and $\alpha<\underline{c}(M,\varphi)-4\varepsilon$.

But for this ε there exists a $\rho_1>0$ such that $\frac{1}{t}\log M_\varepsilon<\varepsilon$ for all $t\geq\rho_1$. Let $\rho=\max(\rho_0,\rho_1)$. Since φ_τ is surjective on Y for all $\tau\geq 0$, there exists $x_{\hat{f}}\in X$ with $\|\hat{f}\|=|||\hat{f}(\varphi_\rho(x\hat{f}))|||$ for all $\hat{f}\in C(Y,\mathbb{C}^n)$ and we have:

$$\begin{aligned}\|(e^{\lambda.p}-\hat{T}(p))\hat{f}||| &\geq |||Q_p(x_{\hat{f}})\hat{f}(\varphi_\rho(x_{\hat{f}}))-e^{\rho.\lambda}\hat{f}(x_{\hat{f}})|||\\ &\geq []Q_p^{-1}(x_{\hat{f}})[]^{-1}.|||\hat{f}(\varphi_\rho(x_{\hat{f}}))|||-e^{\rho.\lambda}\|\hat{f}\|\\ &\geq\left(\frac{1}{M_\varepsilon}e^{\rho(\alpha+2\varepsilon)}-e^{\rho.\alpha}\right)\|\hat{f}\|\\ &\geq\left(\frac{1}{M_\varepsilon}e^{\rho(\alpha+\varepsilon+\frac{1}{\rho}\log M_\varepsilon)}-e^{\rho.\alpha}\right)\|\hat{f}\|\\ &=\left(e^{\rho(\alpha+\varepsilon)}-e^{\rho.\alpha}\right)\|\hat{f}\|\end{aligned}$$

Hence we obtain

$$\|(\lambda-\hat{A})\hat{f}\|\geq\varepsilon\cdot\|\hat{f}\|. \tag{3}$$

Now we have to deal with two cases:

Let $\sigma(\hat{A}) \cap H \neq \emptyset$. Since the topological boundary of $\sigma(\hat{A})$ is contained in $A\sigma(\hat{A})$, by (3), it follows that $H \cap A\sigma(\hat{A}) = \emptyset$. Then $H \subset \sigma(\hat{A}) = R\sigma(\hat{A}) \cup A\sigma(\hat{A})$ or $H \cap \sigma(\hat{A}) = \emptyset$. Thus $H \subset R\sigma(\hat{A}) \subset R\sigma(A)$.

Let $\sigma(\hat{A}) \cap H = \emptyset$ By the assumption there exists $t \geq 0$ such that $Y = \bigcap_{s \geq 0} \varphi_s(X) = \varphi_t(X)$. If we consider the induced operator $(T(t)_|)$ we see that for all $\tilde{f} \in J$

$$T(t)_| \tilde{f}(x) = T(t)\tilde{f}(x) = Q_t(x)\tilde{f}(\varphi_t(x)) = 0,$$

that is $(T(t)_|)$ is nilpotent. Thus

$$\lim_{n \to \infty} \|T(t)_|^n\|^{\frac{t}{n}} = r(T(t)_|) = 0.$$

Since $e^{t\sigma(A_|)} \subset \sigma(T(t)_|)$ we have $\sigma(A_|) = \emptyset$. On the other hand by (2) we have $\sigma(A) \subset \sigma(\hat{A}) \cup \sigma(A_|)$. Thus $\sigma(A) \subset \sigma(\hat{A})$. Consequently $\sigma(A) \cap H = \emptyset$ or $H \subseteq R\sigma(A)$.

This completes the proof.

Suppose that $\varphi(\mathbb{R}_+, x_0) = \varphi([0, \tau], x_0)$ for a $\tau < \infty$ and some $x_0 \in X$. Denote $\tau\colon = \inf\{t > 0\colon \varphi_{\tau+t}(x_0) = \varphi_{\tau+t}(x_0)\} > 0$. Then $\varphi_{\tau+t}(x_0) = \varphi_\tau(x_0)$ iff $\tau + t = \tau + n\tau$ for a $n \in \mathbb{N}$. Hence $\tau \in \inf\{t > 0\colon \varphi_{\tau+t}(x_0) = \varphi_\tau(x_0)\}$ and $\varphi_{\tau+\tau}(x_0) = \varphi_\tau(x_0)$.

Let M be invertible matrix. By [1], VIII.8 there exists $L \in M_n(\mathbb{C})$ such that $e^L = M$. Note that this representation of M is not unique, but $\sigma(L)$ is unique modulo $2\pi i$.

In the light of the above we are giving the following proposition.

Proposition 5. *Let $x_0 \in X$ and $\varphi(\mathbb{R}_+, x_0) = \varphi([0, \tau], x_0)$ for a $\tau < \infty$. Let $\tau\colon = \inf\{t > 0\colon \varphi_{\tau+t}(x_0) = \varphi_\tau(x_0)\} > 0$ and*

$$\hat{H}\colon = \frac{1}{\tau} \log \left(\lim_{n \to \infty} e^{n \int_0^{\frac{\tau+\tau}{n}} M(\varphi_s(x_0))ds} . e^{-n \int_0^{\frac{\tau}{n}} M(\varphi_s(x_0))ds} \right).$$

if $\lambda \in \sigma(\hat{H})$, then $\lambda + \frac{2k\pi i}{\tau}\mathbb{Z} \subset R\sigma(A)$.

Proof. To complete the proof we will need the equation

$$\begin{aligned} T(\tau + \tau)f(x_0) &= Q_{\tau+\tau} f(\varphi_{\tau+\tau}(x_0)) \\ &= \lim_{n \to \infty} e^{n \int_0^{\frac{\tau}{n} + \frac{\tau}{n}} M(\varphi_s(x_0))ds} f(\varphi_{\tau+\tau}(x_0)) \\ &= \lim_{n \to \infty} e^{n \int_0^{\frac{\tau}{n} + \frac{\tau}{n}} M(\varphi_s(x_0))ds} \\ &\quad e^{-n \int_0^{\frac{\tau}{n}} M(\varphi_s(x_0))ds} . e^{n \int_0^{\frac{\tau}{n}} M(\varphi_s(x_0))ds} . f(\varphi_\tau(x_0) \end{aligned}$$

Since

$$e^{\tau\hat{H}} = \lim_{n\to\infty} e^{\int_0^{\frac{\tau}{n}+\frac{\tau}{n}} M(\varphi_s(x_0))ds}.e^{-n\int_0^{\frac{\tau}{n}} M(\varphi_s(x_0))ds},$$

$$(*) \qquad T(\tau+\tau)f(x_0) = e^{\tau\hat{H}}T(\tau)f(x_0).$$

Let $\lambda \in \sigma(\hat{H})$ and $0 \neq \omega \in \mathbb{C}^n$ with $\hat{H}'\omega_0 = \lambda\psi_0$. Foı $\mu \in \mathbb{C}$ we consider the linear form

$$\Psi_\mu(f) =< \int_\tau^{\tau+\tau} e^{\mu t}T(t)f(x_0)dt, \omega_0 > .$$

Let $f \in D(A)$. Then

$$\begin{aligned}
\Psi_\mu((\mu+A)f) &=< \int_\tau^{\tau+\tau} e^{\mu t}T(t)(\mu+A)f(x_0)dt, \omega_0 > \\
&=< e^{(\tau+\tau)\mu}T(\tau+\tau)f(x_0) - e^{\tau_\mu}T(\tau)f(x_0), \omega_0 > \\
&=< e^{(\tau+\tau)\mu}e^{\tau\hat{H}}T(\tau)f(x_0) - e^{\tau_\mu}T(\tau)f(x_0), \omega_0 > \\
&=< e^{(\tau+\tau)\mu}T(\tau)f(x_0), e^{\tau\hat{H}'}\omega_0 > - < e^{\tau_\mu}T(\tau)f(x_0), \omega_0 > \\
&=< e^{(\tau+\tau)\mu}T(\tau)f(x_0), e^{\tau\lambda}\omega_0 > - < e^{\tau_\mu}T(\tau)f(x_0), \omega_0 > \\
&=< e^{(\tau+\tau)\mu}e^{\tau\lambda}T(\tau)f(x_0) - e^{\mu\tau}T(\tau)f(x_0), \omega_0 > \\
&=< (e^{\tau\mu+\tau\lambda}-1)e^{\mu\tau}T(\tau)f(x_0), \omega_0 > \\
&=< (e^{\tau\mu+\tau\lambda}-1) < e^{\mu\tau}T(\tau)f(x_0), \omega_0 >= 0.
\end{aligned}$$

In the case $\Psi_\mu((\mu+A)f) = 0$ iff $\tau\mu + \tau\lambda \in 2\pi\mathbb{Z}$ (, that is then $\mu \in \frac{2\pi i\mathbb{Z}}{\tau} - \lambda$). Then - $\mu \in P\sigma(A')$, that is $\lambda + \frac{2\pi i\mathbb{Z}}{\tau} \in R\sigma(A)$.

For $\mu = \frac{2\pi ik}{\tau}$, $k \in \mathbb{Z}$ we show that $\Psi_\mu \neq 0$

$$\begin{aligned}
\Psi_\mu(f) &=< \int_\tau^{\tau+\tau} e^{\mu.t}Q_t(x_0)f(\varphi_t(x_0))dt, \omega_0 > \\
&=< \int_\tau^{\tau+\tau} < e^{\mu.t}f(\varphi(t(x_0))), Q_t'(x_0)\omega_0 > dt
\end{aligned}$$

Define

$$\omega(t) :\, = Q_t'(x_0)\cdot\omega_0 = \begin{pmatrix} \omega_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \omega_n(t) \end{pmatrix},$$

where $Q_t(x_0) = \lim_{n\to\infty} \exp(n \int_0^{\frac{t}{n}} M(\varphi_\tau(x_0))d\tau)$. Since $Q_t(x_0)$ is invertible for all $t \in [\tau, \tau + \tau]$ we have $w_{(t)} \neq 0$. We consider the map

$$\tilde{f}_{i\varepsilon} : \varphi([\tau, \tau + \tau - \varepsilon], x_0) \to \mathbb{C}$$

$$\varphi_t(x_0) \longmapsto \bar{\omega}_i(t).$$

By Urysohn, there exists a continuous extension $f_{i,\varepsilon} \in (C(X)$ of $\tilde{f}_{i,\varepsilon}$ with $\|f_{i\varepsilon}(x)\| \leq \|\tilde{f}_{i,\varepsilon}\| \leq \sup_{t\in[\tau,\tau+\tau-\varepsilon]} |\omega_i(t)|$. We define $f_\varepsilon \in C(X, \mathbb{C}^n)$ with

$$f_\varepsilon : = \begin{pmatrix} f_{1,\varepsilon} \\ \cdot \\ \cdot \\ \cdot \\ f_{n,\varepsilon} \end{pmatrix} \cdot e^{-\frac{\varepsilon k \pi i}{\tau} \cdot t}$$

for all $t \in [\tau, \tau + \tau - \varepsilon]$ and $k \in \mathbb{Z}$. Thus $\|f_\varepsilon\| \leq \sup_{t\in[\tau,\tau+\tau-\varepsilon]} |||\omega(t)|||$ and

$$\lim_{\varepsilon\to 0} \Psi_\mu(f_\varepsilon) = \int_\tau^\tau \sum_{i=1}^n |\omega_i(t)|^2 dt > 0.$$

It follows that $\Psi_\mu \neq 0$.

1. Definition 6:

A point $x_0 \in X$ is called *periodic point* with *period* $t_0 > 0$, if $x_0 = \varphi_{t_0}(x_0)$ and $x_0 \neq \varphi_t(x_0)$ for all $t \in (0, t_0)$.

Lemma 7. *Let $M \in M_n(\mathbb{C})$. The following conditions are equivalent.*
(a) There exists a $C \geq 1$ with $[]e^{tM}[] \leq C.e^{t.s(M)}$ for all $t \geq 0$.
(b) For all $\lambda \in \sigma(M)$ with $\operatorname{Re}\lambda = s(M)$, the Jordanblock of λ is diagonal.
(b) For all $\lambda \in \sigma(M)$ with $\operatorname{Re}\lambda = s(M)$, λ is a simple pole of $R(., M)$.

Now it is time to give the following proposition.

Proposition 8. *Let $x_0 \in X$ be a nonperiodical point and let the limit $\hat{M} : = \lim_{t\to\infty} M(\varphi_t(x_0))$ exist. If one of the equivalent conditions of Lemma (7) holds for $\hat{M}$, then $s(\hat{M}) + i\mathbb{R} \subseteq \sigma(A)$.*

Proof. Let $\lambda \in \sigma(\hat{M})$ with $\operatorname{Re}\lambda = s(\hat{M})$ and $\nu \in \mathbb{R}$. We show that $\lambda + i\beta \in A\sigma(A')$.

Let $v_0 \in \mathbb{C}^n$ with $|||v_0||| = 1$ and $\hat{M}v_0 = \lambda v_0$. Let $\omega_0 \in \mathbb{C}^n$ with $< v_0, \omega_0 >= |||\omega_0||| = 1$.

For $m, k \in \mathbb{N}$, let us denote the linear form

$$\begin{aligned}\Psi_{mk}(f) := & \frac{1}{m} < \int_0^m e^{-t\cdot(\lambda_i\beta)} T(t) f(\varphi_k(x_0))dt, \omega_0 > \\ = & \frac{1}{m} < \lim_{n\to\infty} \int_0^m e^{-t\cdot(\lambda_i\beta)} e^{\int_0^t M(\varphi_{s+k}(x_0))ds} f(\varphi_{s+k}(x_0))dt, \omega_0 >\end{aligned}$$

for $f \in E$. For $f \in D(A)$ we have

$$\begin{aligned}< (\lambda + i\beta - A)f, \Psi_{m,k} > = & \frac{1}{m} < \lim_{n\to\infty} f(\varphi_k(x_0)) - e^{-m(i\Im(\lambda)+i\beta)} \\ & \cdot e^{n\int_k^{\frac{m}{n}+k}(M(\varphi_s(x_0))-s(\hat{M}))ds} \cdot f(\varphi_{m+k}(x_0)), \omega_0 > .\end{aligned}$$

Clearly for all $m, k \in \mathbb{N}$ the map

$$D(A) \ni f \mapsto < (\lambda + i\beta - A)f, \Psi_{mk} > \in \mathbb{C}$$

is continuous and hence $\Psi_{mk} \in D(A')$.

If we let k tends to infty and use Lemma (7)(a) we get

$$[] \lim_{n\to\infty} e^{n\int_k^{\frac{m}{n}+k}(M(\varphi_s(x_0))-s(\hat{M}))ds} [] \to [] e^{m(\hat{M}-s(\hat{M}))} [] \leq C$$

and for this reason

$$(*) \qquad \overline{\lim}_{k\to\infty} | < (\lambda + i\beta - A)f, \Psi_{mk} > | \leq \frac{1}{m}(1 + C) \cdot |||\omega_0||| = \frac{1}{m}(1 + C)$$

for all $m \in \mathbb{N}$. Now we show that $\overline{\lim}_{k\to\infty} \|\Psi_{mk}\| \geq 1$. Since x_0 is a non periodical point, the map

$$\begin{aligned}\tilde{\alpha}_{mk} : \varphi([0, m + k], x_0) & \to \mathbb{C} \\ \varphi(t, x_0) & \longmapsto e^{i(t-k)(\Im(\lambda)+\beta)}\end{aligned}$$

is well-defined. By Urysohn there exists a continuous extension $\alpha_{mk} \in C(X)$ of $\tilde{\alpha}_{mk}$ with $|\alpha_{mk}(x)| \leq 1$ for all $x \in X$. We denote $f_{mk} \in E$ by $f_{mk}(x) = \alpha_{mk}(x).v_0$, where v_0 is eigenvector of $\hat{M}$ corresponding to λ. Hence $\|f_{mk}\| = 1$ and

$$\begin{aligned}\Psi_{mk}(f_{mk}) = & < \frac{1}{m} \int_0^m e^{-ti\cdot(\Im(\lambda)+\beta)} e^{n\int_k^{k+\frac{t}{n}}(M(\varphi_s(x_0))-s(\hat{M}))ds} \alpha_{mk}(\varphi_{k+t}(x_0)) v_0 dt, \omega_0 > \\ = & \frac{1}{m} < \int_0^m e^{n\int_k^{k+\frac{t}{n}}(M(\varphi_s(x_0))-s(\hat{M}))ds} v_0 dt, \omega_0 > .\end{aligned}$$

Then

$$\lim_{k\to\infty} \Psi_{mk}(f_\alpha) = \frac{1}{m} < \int_0^m e^{t.(\hat{M}-s(\hat{M}))} v_0 dt, \omega_0 >$$
$$= \frac{1}{m} < m \cdot v_0, \omega_0 >= 1.$$

Finally we obtain

$$(**) \qquad \overline{\lim}_{k\to\infty} ||\Psi_{mk}|| \geq 1.$$

By (*) and (**) we can find a sequence $(k(m)) \subset \mathbb{N}$, such that $(\Psi_{m,k(m)})$ is an approximate eigenvector of A' corresponding to $\lambda + i\beta$.

Proposition 9. *Let $x_0 \in X$ and $\varphi_t(x_0) = x_0$ for all $t \geq 0$. If $\lambda \in \sigma(M(x_0))$, then $\lambda \in R\sigma(A)$.*

Proof. If $\lambda \in \sigma(M(x_0))$, then there exists a vector $\omega_0 \in C^n$ such that $M'(x_0)\omega_0 = \lambda\omega_0$. We consider the linear form

$$\Psi(f) : =< f(x_0), \omega_0 > .$$

Then we have by $\varphi_t(x_0) = x_0$ for all $t \geq 0$

$$Af(x_0) = \lim_{t\to 0} t^{-1}(T(t)f(x_0) - f(x_0))$$
$$= \lim_{t\to 0} t^{-1}(\lim_{n\to\infty} n \cdot \frac{t}{n} M(x_0)f(x_0) - f(x_0))$$
$$= M(x_0)f(x_0).$$

Thus for all $f \in D(A)$ we have

$$\Psi((\lambda - A)f) =< (\lambda - A)f(x_0), \omega_0 >$$
$$=< \lambda f(x_0) - M(x_0)f(x_0), \omega_0 >$$
$$=< \lambda f(x_0), \omega_0 > - < f(x_0), M'(x_0)\omega_0 >$$
$$=< \lambda f(x_0), \omega_0 > - < f(x_0), \lambda\omega_0 >= 0$$

Therefore $\lambda \in P\sigma(A')$, that is $\lambda \in R\sigma(A)$. Let $f_0(x) : = \bar{\omega}_0, f_0 \in C(X, \mathbb{C}^n)$.

Then

$$\Psi(f_0) =< \bar{\omega}_0, \omega_0 >= |||\omega_0|||^2 > 0,$$

thus $\Psi \neq 0$. This completes the proof.

2. Example 10.

Take $X: = [-1,1]$ and $E: = C(X,\mathbb{C}^2)$. Consider the continuous flow $\varphi_t(x): = (xe^{-t}), x \in X, t \geq 0$ and the matrix multiplication

$$M(x): = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} \in M_2(\mathbb{C}), \quad x \in X.$$

Clearly all elements of X is a non periodical point for $t > 0$ and

$$\hat{M}: = \lim_{t\to\infty} M(\varphi_t(x)) = \lim_{t\to\infty} \begin{pmatrix} 0 & xe^{-t} \\ -xe^{-t} & 0 \end{pmatrix} = 0.$$

By Corollary 5 in [3] the family of $(T(t))_{t\geq 0}$ defined by

$$T(t)f(x) = \begin{pmatrix} \cos(xe^{-t} - x) & -\sin(xe^{-t} - x) \\ \sin(xe^{-t} - x) & \cos(xe^{-t} - x) \end{pmatrix} f(xe^{-t})$$

for all $f \in E, x \in X$ and $t \geq 0$, is a strong continuous operator semigroup on E. For this semigroup we obtain

$$\omega_o = \lim_{t\to\infty} \sup_{x\in X} s(\frac{1}{t} \begin{pmatrix} 0 & -x(e^{-t} - 1) \\ x(e^{-t} - 1) & 0 \end{pmatrix}) = 0,$$

$\sigma(\hat{M}) = \{0\}$ and $i \cdot \mathbb{R} \subseteq \sigma(A)$.

Acknowledgement: I wish to thank Rainer Nagel and Klaus J. Engel for their guidances.

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Hacettepe University
Faculty of Education
Department of Science Education
06532 Beytepe, Ankara,
Türkiye

Received 18.12.1992